UNIT 10

Quadratics

Introduction

In Unit 6 you saw that many different real-life situations can be modelled by equations of the form

$$y = mx + c,$$

where m and c are constants. The graphs of equations of this form are straight lines, so models based on this type of equation are called *linear models*.

This unit is about equations of the form

$$y = ax^2 + bx + c, \qquad (1)$$

where a, b and c are constants with $a \neq 0$. You will see some real-life situations that can be modelled by this type of equation. The right-hand side of an equation like this is a quadratic expression, so models like these are called *quadratic models*. The graphs of equations of this type have a characteristic curved shape called a *parabola*.

You will also learn some new methods of solving quadratic equations, which can be applied more generally than the method of factorisation that you learned in Unit 9.

The unit starts in Section 1 with some history about a quadratic model developed in the sixteenth century that provided an accurate means of predicting the motion of cannonballs. You will also see in this section that the vehicle stopping distances that you considered in Unit 2 arise from a quadratic model.

Section 2 explores the graphs of equations of form (1), and in Section 3 you will learn some new methods of solving quadratic equations and apply them to some quadratic models.

Section 4 presents a useful algebraic technique, called *completing the square*, that casts new light on the graphs of equations of form (1), and also provides an additional method for solving quadratic equations.

Finally, in Section 5, you will look at several more examples of quadratic models, from areas such as projectile motion, agriculture, economics and geometry. You will see in each case how the model can be used to solve a particular type of problem, called a *maximisation problem*, where the value of one quantity has to be chosen in order to obtain the maximum value of another quantity. An example of a maximisation problem is the problem of choosing what price to charge for goods or services in order to obtain the maximum revenue.

1 Introducing parabolas

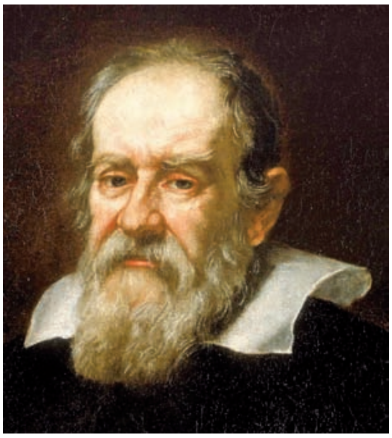
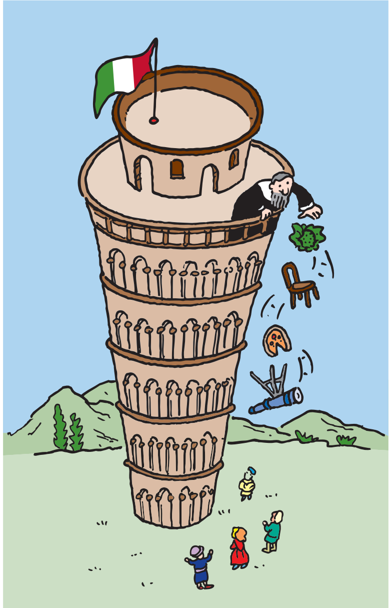


Figure 1 Galileo Galilei (1564–1642) was an Italian astronomer, physicist and mathematician

In the sixteenth century, Galileo Galilei (Figure 1) studied the motion of objects moving under the force of gravity. This work led to important results that allowed him to accurately predict the paths of cannonballs – a useful skill in sixteenth-century Europe. In this section you will learn about some of Galileo's experiments and see how his results can be described using curves called *parabolas*. You will also revisit the topic of vehicle stopping distances, which was discussed in Unit 2.

1.1 Parabolas everywhere



You may have heard the story that Galileo dropped objects of different weights from the top of the Leaning Tower of Pisa, to disprove the commonly held belief that heavy objects fall faster than light objects. It is not known whether this story is true, but Galileo certainly conducted experiments on the motion of falling objects in his laboratory. This was difficult in the sixteenth century, because there were no devices that could accurately measure the short time that it took for an object to fall. Galileo's ingenious solution was to measure, instead, the time that a ball took to roll down a ramp, which is called an *inclined plane* by scientists. He reasoned that the motion of a rolling ball would be similar to that of a falling ball, but slower, enabling him to use the most accurate time-measuring device that he had at his disposal, a water clock.

Figure 2 shows a reconstruction of Galileo's laboratory. In the foreground is the inclined plane – a long sloping plank of wood with a groove on the top edge, down which a bronze ball was rolled.

Figure 2 A reconstruction of Galileo's laboratory at the Deutsches Museum, Munich

Galileo measured the time that the ball took to roll a certain distance from a stationary start, and found that in twice that time it would roll four times the distance, in three times the time it would roll nine times the distance, in four times the time it would roll sixteen times the distance, and so on. In general, if the time was multiplied by n, then the distance was multiplied by n^2. This was true for every angle of the inclined plane that he tried.

Galileo's results are described in his book *Dialogue concerning two new sciences* (1638), as follows:

'We always found that the spaces traversed were to each other as the squares of the times and this was true for all inclinations of the plane.'

From a translation by Henry Crew and Alfonso de Salvio (1914).

Galileo reasoned that this result should also hold for objects in *free fall*, that is, falling under the influence of gravity alone, because as the inclined plane becomes closer and closer to being vertical, the motion of the ball becomes closer and closer to free fall.

Today we have more sophisticated ways of measuring time and distance, and we do not have to 'slow down' the effects of gravity using an inclined plane as Galileo did. We can simply drop a ball and accurately measure the cumulative distance that it has fallen after each second.

Table 1 gives the approximate results that are obtained when this is done. The variables t and d represent the time that the ball has been falling in seconds and the distance that it has fallen in metres, respectively.

Table 1 The distance d (in metres) fallen after time t (in seconds)

t	0	1	2	3	4
d	0	4.9	19.6	44.1	78.4

Figure 3 shows the points in Table 1 plotted on a graph, with time on the horizontal axis and distance on the vertical axis. The points are joined with a smooth curve.

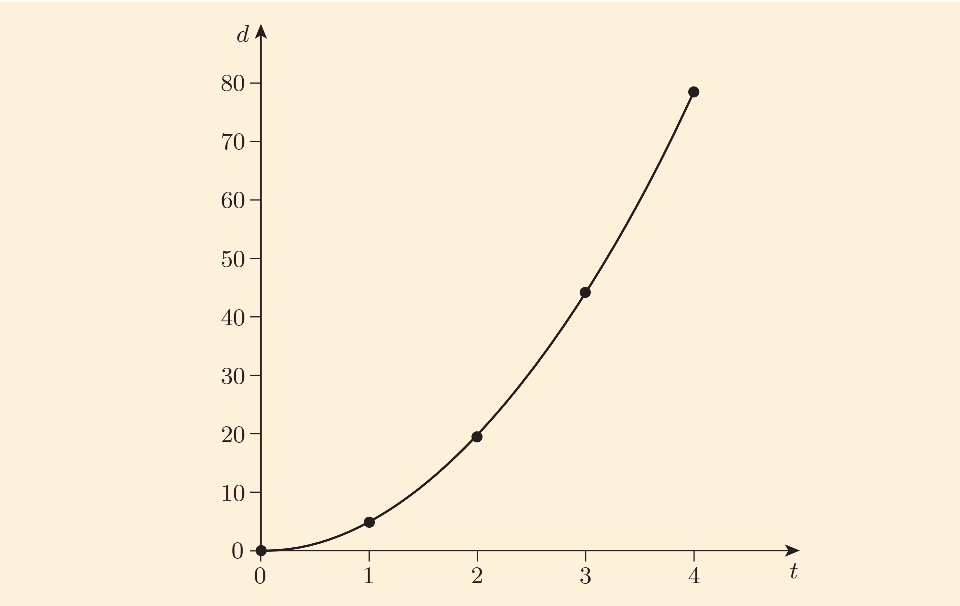


Figure 3 The distance d (in metres) fallen after time t (in seconds)

You met the idea of *direct proportion* in Unit 6.

The distance–time graph in Figure 3 is not a straight line, so the distance fallen by the ball is not proportional to the time that it has been falling – in other words, the ball does not fall at a constant speed. In fact, as you can see, the longer the ball has been falling, the greater the distance that it falls in a given time. For example, between 0 and 1 seconds the ball falls about 5 m, whereas between 1 and 2 seconds it falls about 15 m. So the ball speeds up as it falls – it *accelerates*. The continual increase in the speed of the ball gives the characteristic curved shape of the graph.

Because the graph of the distance fallen by the ball against the time that it has been falling is not a straight line, the relationship between these two quantities cannot be described by an equation of the type that you saw in Unit 6. However, it can be described by a different kind of equation.

You can see from the numbers in Table 1 that the relationship between the time taken and the distance fallen seems to be just as Galileo described:

in 1 second the ball falls 4.9 metres;
in 2 seconds it falls 19.6 metres, which is 4.9×2^2 metres;
in 3 seconds it falls 44.1 metres, which is 4.9×3^2 metres;
in 4 seconds it falls 78.4 metres, which is 4.9×4^2 metres.

So it seems that in general the relationship between d and t is expressed by the equation

$$d = 4.9t^2. \qquad (2)$$

In other words, the distances are proportional to the *squares* of the times, with constant of proportionality 4.9. (This number depends on the fact that the units are metres and seconds.)

From the work of Isaac Newton (Figure 4), a hundred years after Galileo's experiments, it is known that the constant of proportionality in equation (2) is half of the value of the acceleration due to gravity. You learned in Unit 4 that the acceleration due to gravity varies slightly depending on where you are on Earth, but has the approximate value $9.81\,\mathrm{m/s^2}$ (to two decimal places). In this unit the acceleration due to gravity is taken to have the slightly less precise value of $9.8\,\mathrm{m/s^2}$. Elsewhere you might see the even less precise value of $10\,\mathrm{m/s^2}$ used.

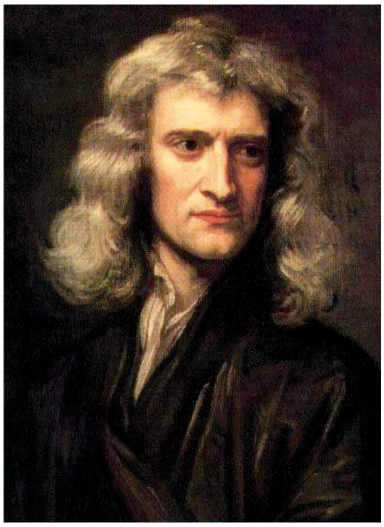


Figure 4 Isaac Newton (1643–1727)

Equation (2) is known as the **free-fall equation**, and it is often written in the form below, in which the acceleration due to gravity is represented by the letter g.

The free-fall equation

The relationship between the distance d fallen by an object and the time t that it has been falling is given by

$$d = \tfrac{1}{2}gt^2,$$

where the constant g is the acceleration due to gravity, which is about $9.8\,\mathrm{m/s^2}$.

If the constant g is expressed in $\mathrm{m/s^2}$, as here, then the variables t and d must be expressed in matching units, namely seconds and metres, respectively.

The curved shape of the graph of the free-fall equation is part of a curve that comes from a special family of curves, called **parabolas**. Some examples of parabolas are shown in Figure 5.

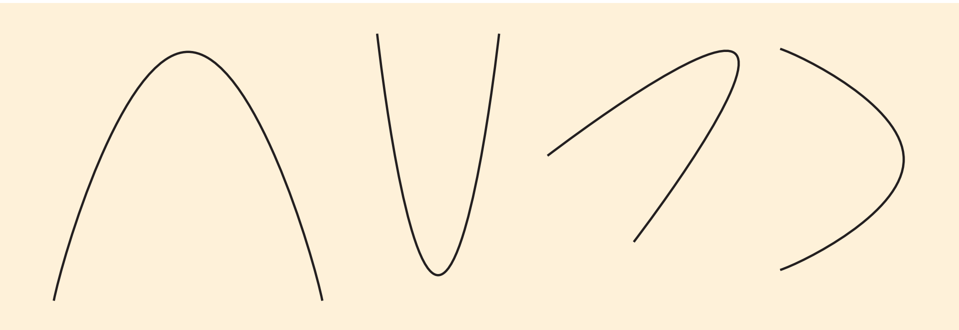


Figure 5 Four parabolas

All the parabolas in the rest of the unit have a *vertical* line of symmetry, like the first two parabolas in Figure 5.

A curve whose shape is all or part of a parabola is said to be **parabolic**. You have probably seen some examples of parabolic curves in your everyday life. A jet of water in a fountain forms a parabola, as does the path of an object that is thrown – you will learn more about this in the next subsection. The reflecting mirror in a torch or car headlight has a cross-section that is a parabola, because this shape of mirror gathers up light from a source and projects it in one direction as a beam. The same shape of reflector also does the opposite; that is, it reflects incoming rays towards a single point. For this reason the cross-section of a satellite dish is a parabola – the dish reflects signals towards a receiver. Figure 6 shows some examples of parabolas and parabolic reflectors.

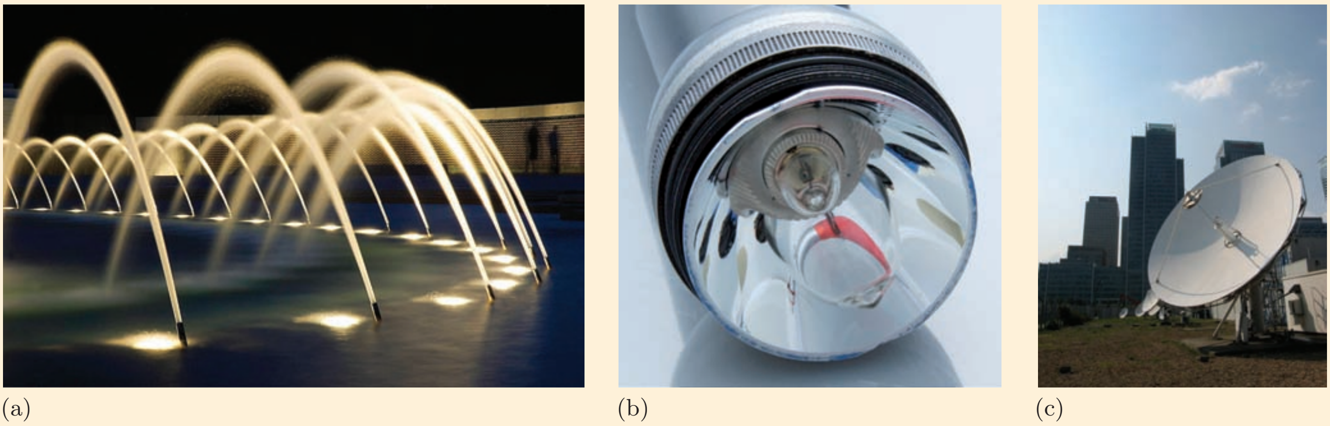


Figure 6 (a) A fountain, (b) the reflector in a torch and (c) a satellite dish

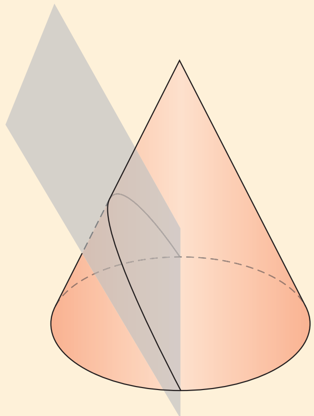


Figure 7 The intersection of a cone and a plane parallel to its side gives a parabola

There are various ways to describe the shape of a parabola, but all the descriptions give the same family of curves. For example, a parabola is the shape obtained when a plane 'parallel' to the side of a cone intersects the cone, as shown in Figure 7. Strictly, the cone should be an 'infinite' cone, as the two ends of a parabola are infinitely long. Alternatively, a parabola is the shape of the graph of any equation of the form

$$y = \text{a quadratic expression in } x \, .$$

That is,

$$y = ax^2 + bx + c, \tag{3}$$

where a, b and c are constants with $a \neq 0$. The free-fall equation has this form, with $a = 4.9$ and $b = c = 0$, and the variables t and d instead of x and y. You will learn much more about parabolas and their equations throughout this unit.

> The word 'parabola' was first used for curves like those in Figure 5 by the Greek geometer and astronomer Apollonius, in around 200 BC, though the shape itself was discovered even earlier. The word means 'juxtaposition' or 'application' in Greek. Later writers thought this word appropriate because the plane shown in Figure 7 can be thought of as being 'juxtaposed to' the cone – parallel to its side.

It is important to appreciate that the free-fall equation is only a *model* for the motion of a falling object. It is accurate if there are no forces other than gravity acting on the object, but in real life other forces do usually affect the motion, notably *air resistance*, which is sometimes called *drag*. Air resistance tends to slow down a falling object. If the object falls a great enough distance, then eventually air resistance will cause its speed to stop increasing and become constant – this constant speed is known as its *terminal velocity*. However, the effects of air resistance are negligible for compact objects falling reasonably short distances, such as the distances in Figure 3 on page 128, and the free-fall equation is a good model for the motion of such objects.

Any model based on an equation of form (3) is called a **quadratic model**.

If you want to know how long it would take for an object to fall a certain distance, then you may be able to read off an approximate answer from the graph of the free-fall equation. For example, the dashed red lines in Figure 8 show that an object would take about 3.8 seconds to fall 70 metres. Alternatively, you can find the answer by substituting into the free-fall equation. You are asked to do this in the first activity.

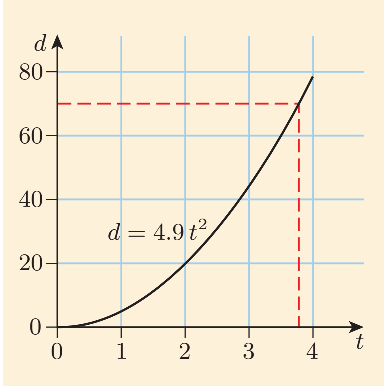


Figure 8 An object takes 3.8 seconds to fall 70 metres

Activity 1 *Using the free-fall equation*

A ball is dropped out of a window at a height of 26 m. By substituting $d = 26$ into the free-fall equation, find the time that the ball takes to reach the ground, to the nearest tenth of a second.

Sometimes, instead of considering the distance that an object has fallen after a particular time, it is more convenient to consider its height above the ground after that time. The height of a falling object at any time is just the difference between its initial height and the distance that it has fallen, as shown in Figure 9.

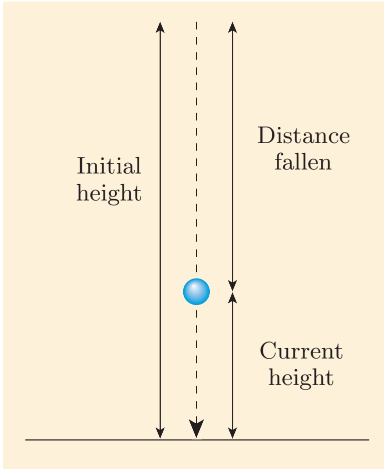


Figure 9 The height of a falling object

For example, consider the ball dropped out of the window in Activity 1. Its initial height is 26 m, and the distance d metres that it has fallen after t seconds is given by the free-fall equation

$$d = 4.9t^2,$$

so its height h metres above the ground after t seconds is given by the equation

$$h = 26 - 4.9t^2. \tag{4}$$

In the next activity you are asked to use Graphplotter to draw the graph of this equation and obtain a result about the motion of the ball.

Graphplotter

Activity 2 *Plotting height against time for the ball*

Use Graphplotter, with the 'One graph' tab selected.

(a) To obtain a graph of the equation

$$y = 26 - 4.9x^2,$$

which is equation (4) with t and h replaced by x and y, respectively, first choose the equation $y = ax^2 + bx + c$ from the drop-down list. Then set $a = -4.9$, $b = 0$ and $c = 26$, by typing these values into the boxes and pressing 'Enter'.

Graphplotter only uses the variables x and y.

Decide on suitable minimum and maximum values for the x- and y-axes, and enter these values in the boxes at the bottom right of Graphplotter. The solution to Activity 1 will help you to choose a suitable maximum value for the x-axis.

(b) Use the Graphplotter graph to find, to the nearest metre, the height of the ball above the ground after 1.2 seconds. To do this, first click the tab for the 'Options' page, and ensure that 'Trace' is ticked and 'Coordinates' and 'y-intercept' are not ticked. Then type 1.2 into the box for the x-coordinate of the cursor (at the bottom left of the graph panel), press 'Enter' and look at the trace coordinates displayed on the graph.

You can use the method in part (b) to find the y-coordinate on any graph corresponding to a given x-coordinate, but not the other way round. You will learn how to go the other way in Subsection 3.1.

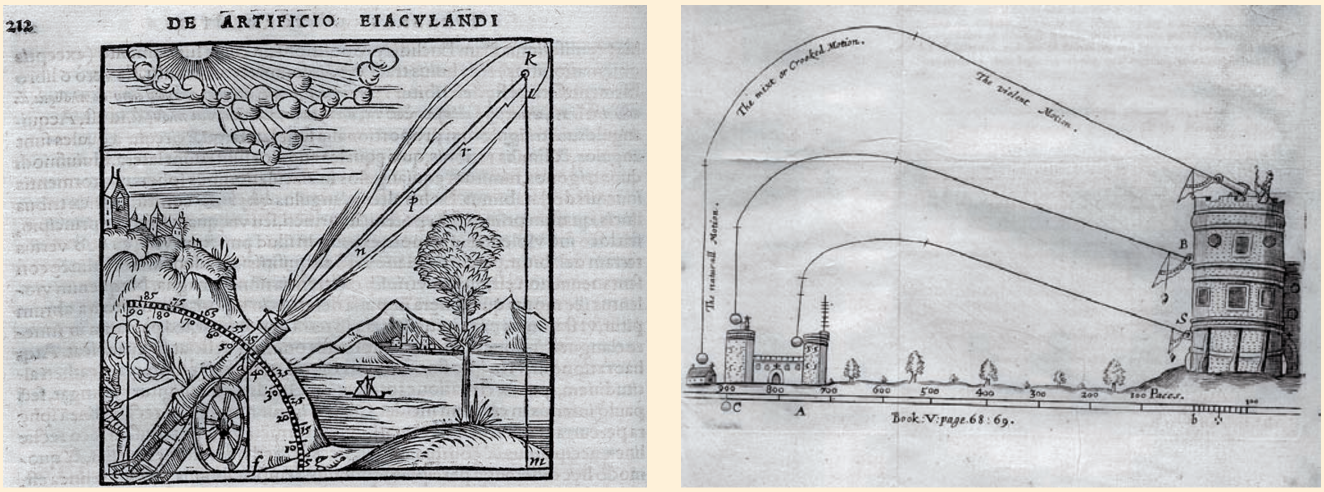
1.2 Projectiles

A **projectile** is an object that is propelled through space by a force that ceases after launch, such as a ball that is thrown, or a cannonball that is fired from a cannon. The **trajectory** of a projectile is the path that it follows. The science of projectiles, especially those fired from firearms, is called **ballistics**.

In medieval times very little accurate information was known about the trajectories of cannonballs, but 'bombardiering', an old name for ballistics, was an increasingly important science for anyone who wanted a military advantage.

An effect due to air turbulence can sometimes cause a spinning object, such as a golf ball, to have a trajectory of roughly the shape shown in Figure 10(b).

Below you can see some guidance given to bombardiers in the sixteenth and seventeenth centuries to help them to hit their targets. In Figure 10(a) the trajectory of a cannonball is shown as consisting of two straight-line segments – one from the muzzle of the cannon to a point marked k on the diagram, and the other vertically downwards from this point to the ground. In Figure 10(b) the trajectory is described as having three phases – the 'violent motion', the 'mixt or crooked motion' and the 'naturall motion'.



(a) (b)

Figure 10 (a) Daniel Santbech, *Problematum astronomicorum et geometricorum sectiones septem* (Basel, 1561). (b) Samuel Sturmy, *The mariners magazine, or Sturmy's mathematical and practical arts* (first edition, 1669; this edition, London, 1684).

Galileo, a Florentine, was appointed to a professorship of mathematics at the University of Pisa in 1589, by Fernando, Duke of Tuscany. Much of Galileo's work on the military applications of mathematics was dedicated to him. The Duke's eldest son, Cosimo II, was taught mathematics by Galileo.

In fact, the author of the book from which Figure 10(b) is taken, Samuel Sturmy, was well behind the times. Galileo had progressed from his work on free-fall motion to experiments on the motion of projectiles, and his results were published in 1638, in the same book in which he discussed his free-fall experiments, *Dialogue concerning two new sciences.*

In his projectile experiments, Galileo continued to use the inclined plane that he had used in his free-fall experiments, but he added a horizontal shelf at the end of the slope, at a height above the floor, so that the ball was launched horizontally into the air. A similar set-up is illustrated in Figure 11.

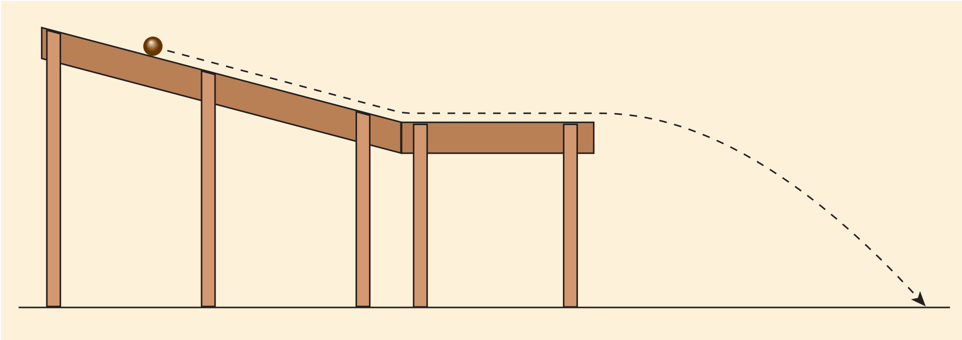


Figure 11 Galileo's projectile experiment

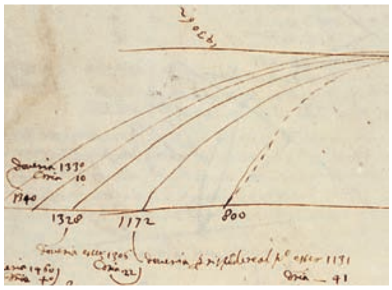


Figure 12 Part of a page from Galileo's notebooks

Galileo rolled an inked ball down the inclined plane and recorded the position on the floor where it landed. By starting the ball at different heights along the inclined plane, he could vary the speed with which the ball left the shelf, resulting in different trajectories. Figure 12 shows some trajectories that Galileo sketched in his laboratory notebook.

Galileo found that the motion of the ball could be understood by thinking about its vertical motion and its horizontal motion separately. At any given time after the ball leaves the shelf, its vertical distance from the end of the shelf, as shown in Figure 13, is determined by the free-fall equation $d = \frac{1}{2}gt^2$, just as if it were falling vertically. At the same time, its horizontal distance from the end of the shelf is determined by the constant speed equation $d = st$, where s is the speed with which the ball leaves the shelf, just as if it were travelling horizontally at that constant speed. The position of the ball at any time is a combination of its horizontal and vertical distances. The different positions of the ball after leaving the shelf give the shape of its trajectory.

Here are Galileo's findings, from a translation of *Dialogue concerning two new sciences*:

'The moving particle, which we imagine to be a heavy one, will on passing over the edge of the plane acquire, in addition to its previous uniform and perpetual motion, a downward propensity due to its own weight; so that the resulting motion which I call projection is compounded of one which is uniform and horizontal and of another which is vertical and naturally accelerated'.

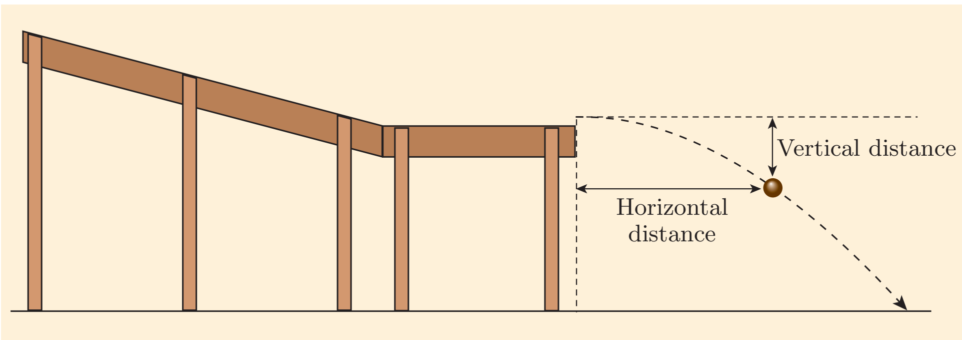


Figure 13 The vertical and horizontal distances of the ball from the shelf

The motion of any projectile launched horizontally, like the inked ball in Galileo's experiments, can be predicted by using the description above. Of course, as with the free-fall equation, this description is just a model, and in real life forces such as air resistance cause the motion of a horizontally-launched projectile to differ to some extent from that predicted by the model.

Figure 14 A cannon from the Alderney being recovered from the sea

Figure 15 A replica of one of the Alderney cannons being fired

In the example below, the model is used to predict the distance travelled by an Elizabethan cannonball fired horizontally. To make the prediction, an estimate for the speed with which a cannonball would have left the muzzle of an Elizabethan cannon is needed.

In 2008, two cannons from an Elizabethan ship known as the Alderney were recovered from the sea off the Channel Islands. The ship is thought to have sunk in 1592, just four years after the defeat of the Spanish Armada, so it is likely that it took part in that great sea battle. A replica was made of one of the cannons, and it was found, much to everyone's surprise, that the maximum muzzle speed was close to the speed of sound, which is about 340 m/s.

Example 1 *Predicting the range of a cannonball*

Suppose that a cannon is fired horizontally from the upper deck of a ship 10 m above the sea, and that the cannonball leaves the muzzle with a speed of 300 m/s.

(a) Calculate the time in seconds that the cannonball will take to hit the sea.

(b) Hence calculate the horizontal distance in metres that the cannonball will travel before hitting the sea.

Give your answers to two significant figures.

Solution

(a) The time t seconds for the cannonball to hit the sea is given by the free-fall equation

$$d = 4.9t^2,$$

where d is the vertical distance in metres between the cannon and the sea, that is, $d = 10$.

Substituting into the equation and solving it gives

$$10 = 4.9t^2$$
$$4.9t^2 = 10$$
$$t^2 = \frac{10}{4.9}$$
$$t = \sqrt{\frac{10}{4.9}} = 1.428\ldots = 1.4 \text{ (to 2 s.f.)}.$$

(The positive square root is taken because the negative one does not make sense in this context.)

So the cannonball will hit the sea after about 1.4 s.

(b) The horizontal distance that the cannonball will travel before hitting the sea is given by the equation $d = st$, where s is the speed with which the cannonball leaves the cannon, and t is the time that it travels before hitting the sea. So $s = 300$ and, from part (a), $t = 1.428\ldots$, where the units are metres per second and seconds, respectively.

Substituting into the equation gives

$$d = 300 \times 1.428\ldots = 430 \text{ (to 2 s.f.)}.$$

So the cannonball will travel for about 430 m before hitting the sea.

Here is a similar activity for you to try.

Activity 3 *Predicting the range of a projectile*

Suppose that a marble is rolled off the edge of a horizontal tabletop with a speed of 0.5 m/s. The tabletop is 0.8 m above the floor.

(a) Calculate the time in seconds that the marble will take to hit the floor.

(b) Calculate the horizontal distance in metres that the marble will travel before hitting the floor.

Give your answers to two significant figures.

A striking consequence of the description of the motion of a horizontally-launched projectile that you saw on page 133 is that if you fire a bullet horizontally from a gun across unobstructed level ground, and drop a bullet from the same height at the same time, then the two bullets will hit the ground at the same time.

Galileo realised that the trajectory of a horizontally-launched projectile predicted by his model is parabolic. To see why this is, consider, for example, the trajectory of the marble in Activity 3. Think of the trajectory as a curve on a graph, as shown in Figure 16.

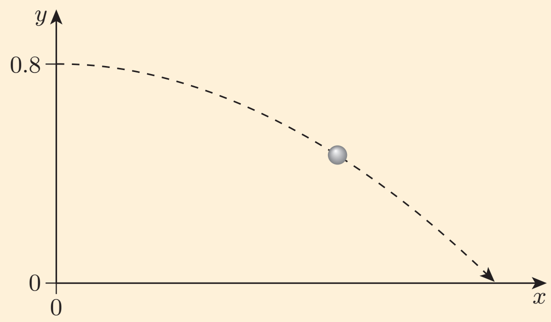


Figure 16 The trajectory of the marble viewed as a graph

For each point on the curve, the x-coordinate is the horizontal distance travelled by the marble after a particular time, and the y-coordinate is its height above the floor at that time, where both measurements are in metres.

Since the initial horizontal speed of the marble is 0.5 m/s, after t seconds the marble has travelled the horizontal distance $0.5t$ metres. The height of the marble at this time is the height of the table minus the distance that the marble has fallen, that is, $0.8 - 4.9t^2$ metres. So, at time t,

$$x = 0.5t \quad \text{and} \quad y = 0.8 - 4.9t^2.$$

Let's express y in terms of x, so that we can see whether this gives a formula of the form

$$y = \text{a quadratic expression in } x \; .$$

You can express y in terms of x by first expressing t in terms of x and then substituting this expression into the equation for y. Since $x = 0.5t$, we have $t = x/0.5$, that is, $t = 2x$. Substituting this into the equation for y gives

$$y = 0.8 - 4.9(2x)^2.$$

Simplifying gives

$$y = 0.8 - 19.6x^2.$$

This formula is of the form

$$y = ax^2 + bx + c,$$

with $a = -19.6$, $b = 0$ and $c = 0.8$. So the curve is indeed part of a parabola. Any trajectory given by Galileo's model for the motion of a horizontally-launched projectile can be shown to be parabolic in the same way.

Galileo went on to look at projectiles launched at various angles, which allowed him to make predictions about the trajectories of cannonballs fired at different angles. He found that the trajectories of projectiles were always parabolic, and by the beginning of the eighteenth century this knowledge was being incorporated into military handbooks, as Figure 17 shows.

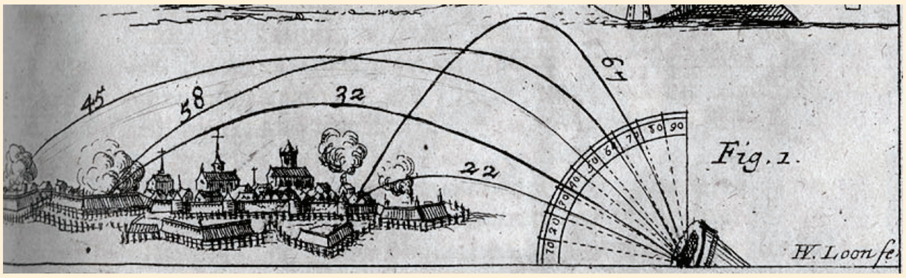


Figure 17 Nicolas Bion, *Traitè de la construction et des principaux usages des instrumens de mathematique* (Paris, 1709)

The diagram in Figure 17 indicates that the maximum range of a cannon is achieved when the cannonball is fired at an angle of 45 degrees, and that there are two possible angles that can be used to hit any closer target.

1.3 Stopping distances

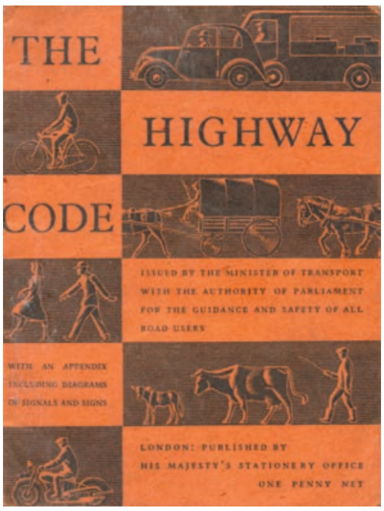


Figure 18 The cover of the third edition (1946) of the Highway Code

In this subsection you will see that the stopping distances given in the Highway Code, which were considered in Unit 2, arise from a quadratic model.

Stopping distances were first included in the third edition of the Highway Code, published in 1946, and they remained essentially unchanged in all subsequent editions, at least up to the time of writing of MU123. Imperial units were predominant in 1946, so the stopping distances were given in feet, with the vehicle speeds in miles per hour. As now, each stopping distance was made up of two parts, the thinking distance and the braking distance, as you can see from Table 2.

Table 2 The thinking, braking and stopping distances published in the third edition (1946) of the Highway Code

Speed (mph)	Thinking distance (feet)	Braking distance (feet)	Overall stopping distance (feet)
10	10	5	15
20	20	20	40
30	30	45	75
40	40	80	120
50	50	125	175

The thinking distances in Table 2 are based on a linear model. If s is the vehicle speed in miles per hour and T is the thinking distance in feet, then, as you can see, T is given by the simple formula

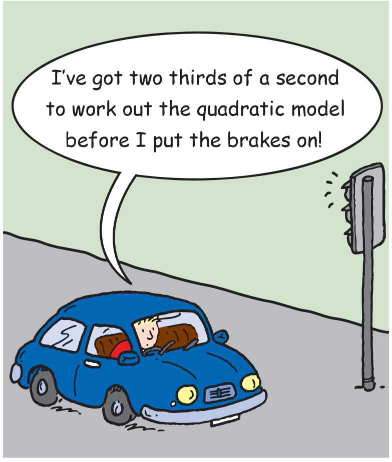
$$T = s.$$

The braking distances are based on a quadratic model. If s is the vehicle speed in miles per hour, as before, and B is the braking distance in feet, then B is given by the formula

$$B = \tfrac{1}{20}s^2. \qquad (5)$$

Activity 4 *Checking the formula for the braking distances*

Choose one of the speeds given in Table 2, and calculate the corresponding braking distance given by formula (5). Check that your answer is the same as the braking distance given in the table.



The expressions for the thinking distances and the braking distances can be added together to give a formula for the overall stopping distances. This formula is

$$D = \tfrac{1}{20}s^2 + s, \qquad (6)$$

where s is the vehicle speed in miles per hour and D is the overall stopping distance in feet. So the overall stopping distances are also given by a quadratic model and hence they lie on a parabolic curve.

The graph of formula (6) is shown in Figure 19, with the points given by the numbers in Table 2 marked on the curve.

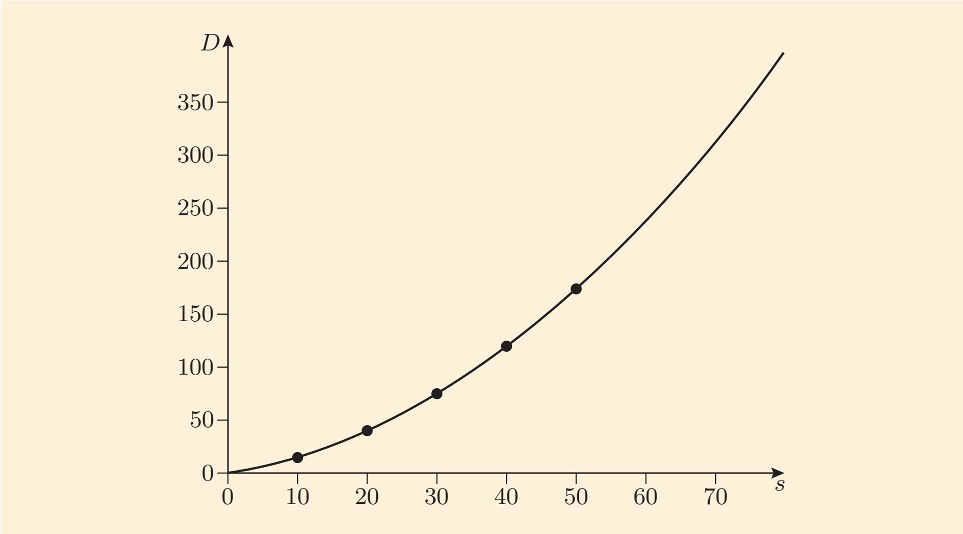


Figure 19 The graph of the formula $D = \frac{1}{20}s^2 + s$

Figure 20 shows the stopping distances given in the 2007 version of the Highway Code. They are the same as those in Table 2, with small differences arising from rounding when the thinking and braking distances are converted to metres. The 2007 version gives a different range of speeds – it covers 20 mph to 70 mph, whereas the 1946 version covers 10 mph to 50 mph – but the stopping distances for 60 mph and 70 mph are also given by formula (6), as you might like to check.

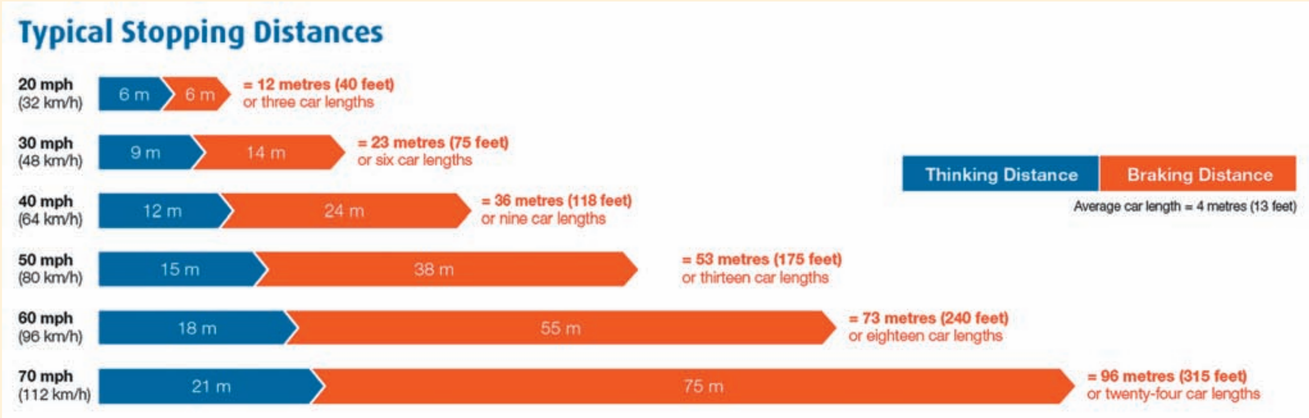


Figure 20

In this section you have read about the experiments that Galileo carried out in order to investigate the motion of objects in free fall and the trajectories of projectiles, and you have learned more about vehicle stopping distances. You saw that the distance–time graph of free-fall motion, the trajectory of a projectile and the graph that models vehicle stopping distances are all parabolic curves.

2 Graphs of quadratic functions

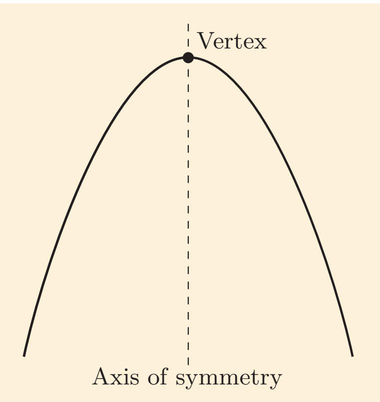


Figure 21 The axis of symmetry and vertex of a parabola

In Section 1 you learned that the graph of an equation of the form

$$y = ax^2 + bx + c, \tag{7}$$

where a, b and c are constants with $a \neq 0$, has a shape called a *parabola*. In this section you will explore the different shapes of parabolas obtained by plotting graphs of this form, and the different positions of these parabolas relative to the axes.

One feature that you will see is that every parabola has a line of symmetry, as shown in Figure 21. Another name for a line of symmetry is an **axis of symmetry**, and this is the phrase that is usually used when discussing parabolas. The axis of symmetry of a parabola cuts the parabola at exactly one point. This point is called the **vertex** of the parabola, also shown in Figure 21.

This section also uses the following terminology, which you met in Unit 6. Whenever you have an equation that expresses one variable in terms of another variable, you can think of it as a rule that takes an input value and produces an output value. For example, if the equation is $y = x^2$, then inputting $x = 3$ gives the output $y = 9$, inputting $x = -1$ gives $y = 1$, and so on. A rule that takes input values and produces output values like this is called a **function**.

A function whose rule is of the form $y = ax^2 + bx + c$, where a, b and c are constants with $a \neq 0$, is called a **quadratic function**. So this section is about the graphs of quadratic functions.

2.1 Graphs of equations of the form $y = ax^2$

We begin by looking at the graphs obtained when the constants b and c in equation (7) are both zero, that is, when the equation has the form

$$y = ax^2, \quad \text{where } a \neq 0.$$

For example, the equations

$$y = 2x^2, \quad y = x^2 \quad \text{and} \quad y = -x^2$$

are of this form, with a equal to 2, 1 and -1, respectively.

We begin by looking at the simple equation

$$y = x^2.$$

You can plot a graph of this equation by constructing a table of values, in the way that you saw in Unit 6. You choose some appropriate values for x, and calculate the corresponding values of y by substituting into the equation. You can choose both positive and negative values of x, and Table 3 shows some results obtained in this way.

Table 3 A table of values for the equation $y = x^2$

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Notice the symmetry in the y-values in Table 3: the values decrease until they reach 0, and then they increase again in the same steps. This happens because $(-x)^2 = x^2$, for any x.

If you plot the seven points given by Table 3 and join them with a smooth curve, then you obtain the graph in Figure 22.

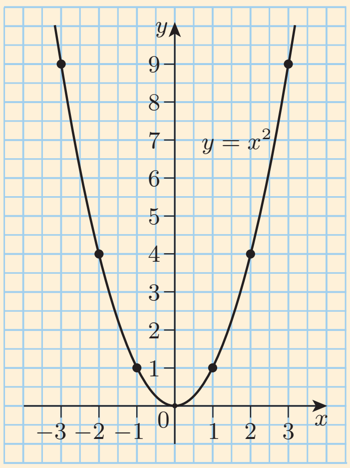


Figure 22 The graph of $y = x^2$

Notice that the curve in Figure 22 has a line of symmetry (the y-axis), as expected. This is because of the symmetry of the y-values.

In the next activity you are asked to plot the graph of another simple quadratic function.

Activity 5 *Plotting the graph of* $y = -x^2$

(a) By constructing a table of values, plot the graph of the equation $y = -x^2$.

(b) Compare your graph to the graph of $y = x^2$ in Figure 22, and write a sentence or two to explain what you see.

Remember that $-x^2$ means the negative of the square of x, not the square of the negative of x. For example,

$$-3^2 = -9 \quad \text{(not 9)}.$$

(A minus sign indicating a negative has the same precedence in the BIDMAS rules as a minus sign indicating subtraction.)

In Activity 5 you saw that the graphs of $y = ax^2$ when $a = 1$ and when $a = -1$ are exactly the same shape, but mirror images of each other, reflected in the x-axis. This is because changing a from 1 to -1 in the equation $y = ax^2$ changes all the y-values to their negatives, as you can see if you compare Table 3 to the table in the solution to Activity 5.

The same thing happens for any value of a: if you change the value of a to its negative, then all the y-values obtained from the equation $y = ax^2$ change to their negatives, so the parabola changes to its mirror image, reflected in the x-axis.

In the next activity you are asked to use Graphplotter to investigate the graph of the equation $y = ax^2$ for some more values of a, both positive and negative.

Graphplotter

Activity 6 *Exploring the graph of* $y = ax^2$

(a) First use Graphplotter with the 'One graph' tab selected. Choose the equation $y = ax^2 + bx + c$ from the drop-down list. Set b and c to 0, and keep them set to 0 throughout this activity, since the aim is to explore the graph of $y = ax^2$.

Set a to 1, and check that you obtain the graph shown in Figure 22 on page 139. Now use the slider to increase the value of a, and then to decrease it again, but only within positive values of a. What is the effect on the shape of the graph as you change the value of a?

(b) Now choose the 'Two graphs' tab in Graphplotter. Choose the equation $y = ax^2 + bx + c$ from both drop-down lists, and check that b and c are set to 0 for both graphs.

Set a to 1 and -1 for the first and second graphs, respectively, and check that the graphs are reflections of each other in the x-axis, as expected. Try some other pairs of values of a that are negatives of each other, and check that the graphs are as you expect. You might also like to use the slider to see how the graph changes as a changes within the negative values.

The magnitude of a number is sometimes called its *size*, and this was the word used in Unit 6. For example, the magnitude of -3 is 3, and the magnitude of 3 is also 3.

You should have found in Activity 6 that the value of a appears to affect how narrow the graph of $y = ax^2$ is. More specifically, it is the *magnitude* of a that affects the width of the graph – the **magnitude** of a number is its value without its negative sign, if it has one. The larger the magnitude of a, the narrower the parabola becomes.

To see why this is, think of a point, other than $(0, 0)$, on the graph of $y = x^2$, say. Now imagine changing the graph to make it the graph of $y = 2x^2$. The point with the same x-value now has a y-value twice as large, so it has moved up. All the points on the parabola except $(0, 0)$ move up

in this way. The further away they are from the vertex $(0, 0)$, the larger their y-value is to start with, so the more they move up. This has the effect of making the parabola more narrow. Similarly, if you change $y = -x^2$ to $y = -2x^2$, then the points move down, so again the parabola becomes narrower. The axis of symmetry of the graph is not affected.

Because the point $(0, 0)$ on the parabola does not move when you change the value of a, the vertex of the parabola does not change. So for all values of a, the vertex of the graph of $y = ax^2$ is at the origin.

Another property that you saw in Activity 6 is that, as expected, if a is positive, then the graph of $y = ax^2$ is the same way up as the graph of $y = x^2$, while if a is negative, then it is the other way up. A parabola that is the same way up as the graph of $y = x^2$ is called a **u-shaped** parabola, while one that is the other way up is called an **n-shaped** parabola. These two possibilities are illustrated in Figure 23.

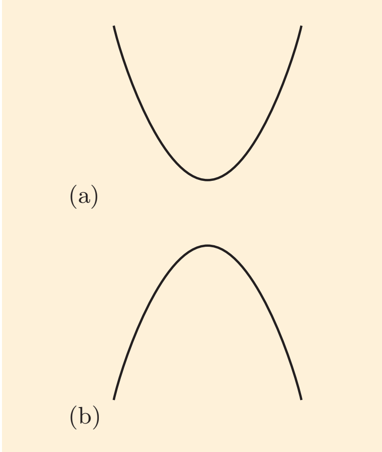


Figure 23 (a) A u-shaped parabola (b) An n-shaped parabola

Here is a summary of what you have learned about the graphs of equations of the form $y = ax^2$ in this subsection.

> ***The graph of the equation*** $y = ax^2$
>
> The vertex is $(0, 0)$.
>
> If a is positive, then the graph is u-shaped.
>
> If a is negative, then the graph is n-shaped.
>
> The larger the magnitude of a, the narrower the parabola.

2.2 Graphs of equations of the form $y = ax^2 + bx + c$

In this subsection you will explore the graphs of general quadratic functions.

In the first activity you are asked to explore the graphs that are obtained when the coefficient c in the equation $y = ax^2 + bx + c$ is zero. That is, you will explore the graphs of equations of the form

$$y = ax^2 + bx.$$

Graphplotter

Activity 7 *Graphs of equations of the form* $y = ax^2 + bx$

Use Graphplotter, with the 'One graph' tab selected.

(a) Choose the equation $y = ax^2 + bx + c$ from the drop-down list. Set $a = 1$, $b = 0$ and $c = 0$, and check that you obtain the graph of $y = x^2$, as expected.

(b) Now explore the effect of changing the value of b. Notice that although the position of the vertex changes, the following features of the graph remain the same:

- its shape
- the fact that its axis of symmetry is vertical
- which way up it is
- the fact that it goes through $(0, 0)$.

(c) Repeat part (b) for one or two other values of a.

In Activity 7 you should have found that for all values of a and b that you tried, the graph of $y = ax^2 + bx$ is exactly the same as the graph of $y = ax^2$, but shifted to a different position relative to the axes. The word 'shifted' here means that the parabola is just slid to a new position – it is not rotated in any way, so its axis of symmetry remains vertical.

You should also have found that for all values of a and b that you tried, the graph of $y = ax^2 + bx$ passes through the point $(0, 0)$. This is because substituting $x = 0$ in the equation $y = ax^2 + bx$ gives $y = 0$.

In the next activity you are asked to look at the effect of changing the value of the constant term c in the equation $y = ax^2 + bx + c$.

Graphplotter

Activity 8 *Graphs of equations of the form* $y = ax^2 + bx + c$

Use Graphplotter, with the 'One graph' tab selected.

Make sure that 'y-intercept' in the Options page is ticked. This causes the coordinates of the point where the graph crosses the y-axis to be displayed.

(a) Choose the equation $y = ax^2 + bx + c$ from the drop-down list. Set $a = 1$, $b = 0$ and $c = 0$, and check that you obtain the graph of $y = x^2$, as expected.

(b) Now set $c = 1$. What is the effect on the graph? In particular, what is the point where it crosses the y-axis?

(c) Choose some other values of c (both positive and negative) and repeat part (b) for each of these values.

(d) Repeat parts (b) and (c) for some other values of a and b, and describe what you find.

In Activity 8 you should have found that for any values of a, b and c, the graph of $y = ax^2 + bx + c$ is exactly the same as the graph of $y = ax^2 + bx$, except that it is shifted vertically up or down, so that it crosses the y-axis at $(0, c)$ instead of $(0, 0)$.

This is because adding the constant c to the right-hand side of the equation $y = ax^2 + bx$ just changes all the y-values by c units, which causes the graph to move up or down.

So, from what you have seen in Activities 7 and 8, it seems that for all values of a, b and c, the graph of $y = ax^2 + bx + c$ is exactly the same shape as the graph of $y = ax^2$, but shifted horizontally and/or vertically to a different position relative to the axes. This is indeed the case, and you will see why later in the unit.

This means that the only possible basic shapes of parabolas are the shapes of the graphs of the form $y = ax^2$. You have seen that these all have a vertical axis of symmetry and differ in how wide they are.

In particular, the graph of the equation $y = ax^2 + bx + c$ is always a u-shaped or n-shaped parabola. If a is positive, then it is u-shaped; if a is negative, then it is n-shaped.

Here is a summary of what you have learned in this subsection about the graphs of quadratic functions.

The graph of the equation $y = ax^2 + bx + c$

If a is positive, then the graph is u-shaped.

If a is negative, then the graph is n-shaped.

The graph has the same shape as the graph of $y = ax^2$, but shifted.

The graph crosses the y-axis at $(0, c)$.

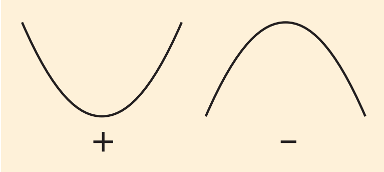


Figure 24 A smile-shaped parabola goes with a positive coefficient of x^2, and a frown-shaped parabola goes with a negative coefficient of x^2

Figure 24 suggests a way to remember the relationship between the sign of the coefficient of x^2 and whether the graph is u-shaped or n-shaped.

2.3 The intercepts of a parabola

You met the idea of the *intercepts* of a graph in Unit 6. An ***x*-intercept** of a graph is a value where it crosses or touches the x-axis. In other words, it is a value of x for which $y = 0$. Similarly, a ***y*-intercept** of a graph is a value where it crosses or touches the y-axis. That is, it is a value of y for which $x = 0$. For example, the x-intercepts of the parabola shown in Figure 25 are -2 and 5, and the y-intercept is -10.

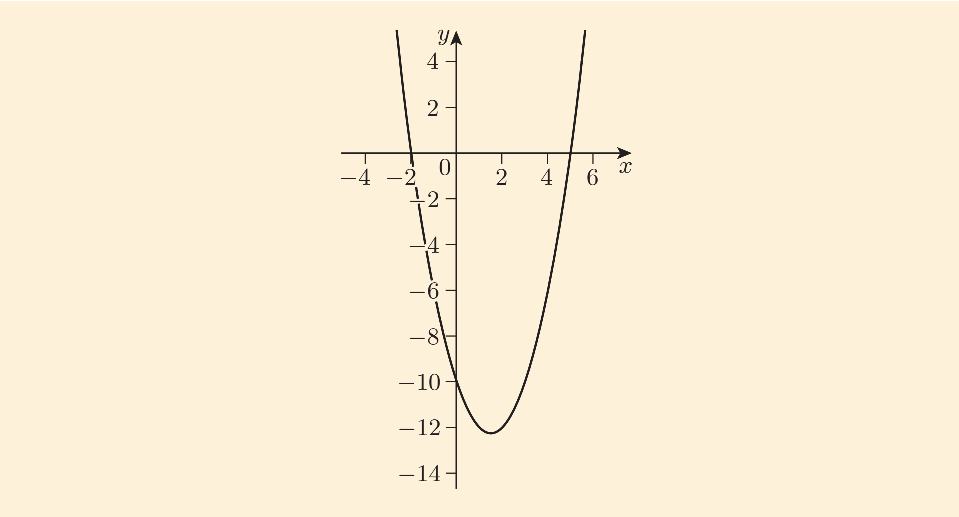


Figure 25 The graph of $y = x^2 - 3x - 10$

The graph of a quadratic function can have two, one or zero x-intercepts, depending on its position relative to the x-axis. The three possibilities are shown in Figure 26, for u-shaped parabolas.

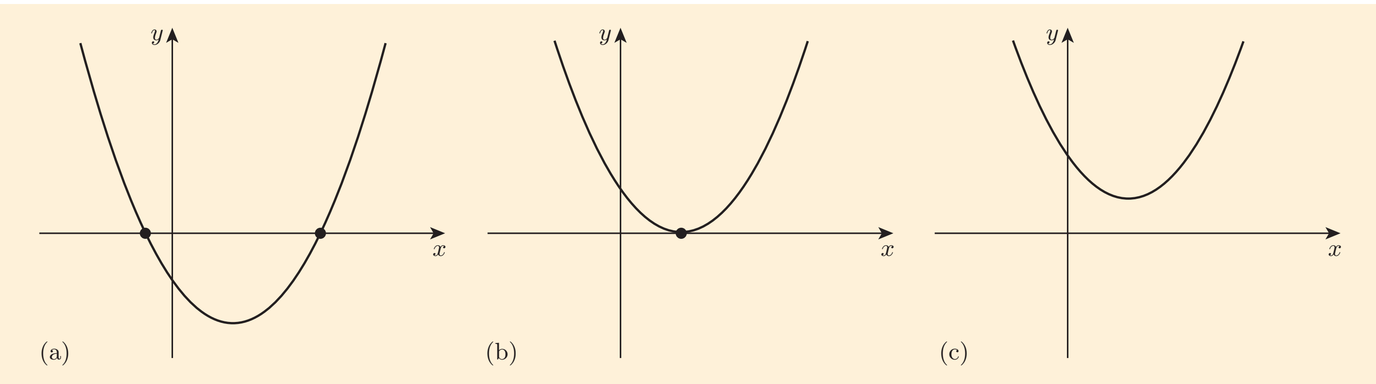


Figure 26 The graph of $y = ax^2 + bx + c$ can have two, one or zero x-intercepts

There is always exactly one y-intercept, because there is exactly one value of y for each value of x, including $x = 0$.

You can use the equation of a parabola to find its intercepts.

Finding the y-intercept

To obtain the y-intercept of a parabola, you substitute $x = 0$ into its equation. For example, consider the parabola with equation

$$y = x^2 - 3x - 10.$$

Substituting $x = 0$ into this equation gives $y = -10$, so the y-intercept of this parabola is -10, as shown in Figure 25.

Notice that whatever the values of a, b and c, if you substitute $x = 0$ into the equation $y = ax^2 + bx + c$, then you obtain $y = c$. So the y-intercept of the graph is always the value c, as was observed in the previous subsection.

Finding the x-intercepts

To obtain the x-intercepts of a parabola, you substitute $y = 0$ into the equation. For example, consider again the parabola with equation

$$y = x^2 - 3x - 10.$$

Substituting $y = 0$ gives

$$0 = x^2 - 3x - 10,$$

so the x-intercepts of the parabola are the solutions of this quadratic equation. The quadratic expression on the right-hand side factorises as $(x + 2)(x - 5)$, so the x-intercepts are -2 and 5, as shown in Figure 25.

Finding the x-intercepts of a parabola always involves solving a quadratic equation. If you cannot solve the equation by factorisation, then you may be able to solve it by using the *quadratic formula*, which you will meet in Section 3.

If the quadratic equation has only one solution, then the parabola has only one x-intercept. If it has no solutions at all, then the parabola has no x-intercepts. In Section 3 you will learn how to tell from the coefficients of a quadratic equation how many solutions it has.

You will have a chance to practise finding the intercepts of parabolas in the next subsection.

2.4 Sketch graphs of quadratic functions

When you are dealing with a quadratic function, it can be useful to have an idea of what its graph looks like. Often you do not need an accurate plot, but just a quick sketch showing some of the main features. The features that you would usually show on such a sketch are as follows:

- whether it is u-shaped or n-shaped
- its intercepts
- its axis of symmetry
- its vertex.

In general, a sketch graph of a function should show its general shape and features such as its intercepts and any points where the curve reaches a maximum or minimum.

You have already seen how to determine whether a parabola is u-shaped or n-shaped from its equation, and how to find its intercepts.

You can find the axis of symmetry by using the fact that this line lies halfway between the x-intercepts, or passes through the single x-intercept if there is only one. You will see later in this subsection how you can find the axis of symmetry if there are no x-intercepts.

The equation of the axis of symmetry tells you the x-coordinate of the vertex, and you can substitute that into the equation of the parabola to find the corresponding y-coordinate.

The next example shows you how you might go about producing a sketch graph of a quadratic function.

Example 2 *Sketching the graph of a quadratic function*

Tutorial clip

This question is about the parabola

$$y = -x^2 + 2x + 8.$$

(a) State whether the parabola is u-shaped or n-shaped, and find its intercepts.

(b) Find the equation of the axis of symmetry, and the coordinates of the vertex.

(c) Sketch the parabola.

Solution

(a) The coefficient of x^2 is negative, so the graph is n-shaped.

Putting $x = 0$ gives $y = 8$, so the y-intercept is 8.

Putting $y = 0$ gives

$$0 = -x^2 + 2x + 8.$$

Multiply through by -1 to make factorising easier.

$$0 = x^2 - 2x - 8$$
$$(x + 2)(x - 4) = 0$$
$$x + 2 = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = -2 \quad \text{or} \quad x = 4$$

So the x-intercepts are -2 and 4.

(b) The axis of symmetry is halfway between the x-intercepts.

The number halfway between the x-intercepts is

$$\frac{-2+4}{2} = \frac{2}{2} = 1,$$

so the axis of symmetry is the line with equation $x = 1$.

To find the number halfway between two numbers, add them together and divide by 2. In other words, calculate their mean.

The vertex is on the axis of symmetry.

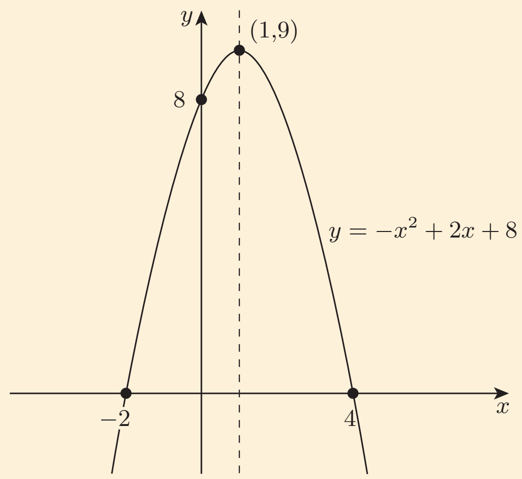
Hence the x-coordinate of the vertex is 1.

Substituting $x = 1$ into the equation of the parabola gives

$$y = -(1)^2 + 2 \times 1 + 8 = -1 + 2 + 8 = 9.$$

So the vertex is $(1, 9)$.

(c) Plot the intercepts and the vertex, and draw the axis of symmetry. Hence sketch the parabola and label it with its equation. Indicate the values of the intercepts and the coordinates of the vertex.



Here is a summary of how to sketch the graph of a quadratic function.

Strategy *To sketch the graph of a quadratic function*

1. Find whether the parabola is u-shaped or n-shaped.
2. Find its intercepts, axis of symmetry and vertex.
3. Plot the features found, and hence sketch the parabola.
4. Label the parabola with its equation, and make sure that the values of the intercepts and the coordinates of the vertex are indicated.

You can practise drawing sketch graphs of quadratic equations in the next two activities. Try to draw each parabola smoothly through the points that you have plotted, and symmetrically on each side of the axis of symmetry. Sometimes finding and plotting one or two extra points on the parabola can help you to draw a good sketch.

As when you draw straight-line graphs, when you sketch parabolas it is usually best to use equal scales on the axes, unless that makes the graph hard to draw or interpret, in which case you should use different scales.

Activity 9 *Sketching the graph of a quadratic function*

Use the strategy above to draw a neat sketch of the graph of the equation $y = x^2 + 5x - 6$.

As you have seen, to find the x-intercepts of a parabola you need to solve a quadratic equation. If you obtain just one solution, then the graph has just one x-intercept. In this case, the single point where the graph touches the x-axis is the vertex, and the vertical line through this point is the axis of

symmetry. You can use this information, together with the y-intercept, to sketch the graph in the usual way. Try this in the next activity.

Activity 10 *Sketching the graph of a quadratic function with one x-intercept*

Sketch the graph of the equation $y = 9x^2 - 6x + 1$.

You have seen that you can often find the axis of symmetry of a parabola by using the fact that it lies halfway between the x-intercepts. An alternative method is to use the formula below. This formula can be used when the parabola has no x-intercepts, and you might prefer to use it in other cases too.

A formula for the axis of symmetry of a parabola

The axis of symmetry of the parabola with equation $y = ax^2 + bx + c$ is the line with equation

$$x = -\frac{b}{2a}.$$

For example, to use the formula to work out the axis of symmetry of the parabola $y = -x^2 + 2x + 8$, which was considered in Example 2 on page 145, you substitute $a = -1$ and $b = 2$, which gives

$$x = -\frac{2}{2 \times (-1)}; \quad \text{that is,} \quad x = 1.$$

This is the same line as was found in Example 2.

To see why the formula works, consider the equation

$$y = ax^2 + bx + c,$$

where a, b and c are constants with $a \neq 0$. You know that the graph of this equation is the same as the graph of

$$y = ax^2 + bx,$$

except that it is shifted vertically. So the two graphs have the same axis of symmetry. The x-intercepts of the second graph can be found by factorisation:

$$\begin{aligned} &ax^2 + bx = 0 \\ &x(ax + b) = 0 \\ &x = 0 \quad \text{or} \quad ax + b = 0 \\ &x = 0 \quad \text{or} \quad x = -\frac{b}{a}. \end{aligned}$$

The value halfway between 0 and $-\frac{b}{a}$ is

$$\frac{1}{2}\left(0 + \left(-\frac{b}{a}\right)\right) = -\frac{b}{2a},$$

so the axis of symmetry is the line $x = -\frac{b}{2a}$.

In the next activity you are asked to sketch the graph of a quadratic equation that has no x-intercepts.

Activity 11 *Sketching a quadratic function with no x-intercepts*

Find the y-intercept, axis of symmetry and vertex of the graph of the equation

$$y = x^2 + 2x + 3,$$

and hence sketch the graph.

In this section you have seen that the graph of an equation of the form $y = ax^2 + bx + c$, where a, b and c are constants with $a \neq 0$, is always a u-shaped or n-shaped parabola. The sign of the constant a tells you whether the parabola is u-shaped or n-shaped, and the magnitude of a determines how wide it is. The y-intercept of the graph is c. The position of the vertex depends on the values of all three coefficients, and you will find out more about this later in the unit.

You have also learned how to sketch the graphs of quadratic functions.

3 Solving quadratic equations

In Unit 9 you saw that you can often use factorisation to find the solutions of a quadratic equation. However, sometimes it is not possible to factorise a quadratic expression in the way that you saw in Unit 9, and even if there is such a factorisation, it can be hard to find it.

In this section you will see two more ways to find the solutions of a quadratic equation, which can be used when factorisation is difficult. You will also learn how to tell how many solutions a quadratic equation has, from the values of its coefficients. You will meet yet another way to solve quadratic equations in Section 4.

The section also includes some real-life problems in which solving a quadratic equation is helpful.

3.1 Solving quadratic equations graphically

If you need only *approximate* values for the solutions of a quadratic equation $ax^2 + bx + c = 0$, then one way to find them is to obtain a fairly accurate graph of the corresponding quadratic function $y = ax^2 + bx + c$ and read off the values of x when $y = 0$, that is, the x-intercepts.

For example, consider the quadratic equation

$$x^2 + 3x - 6 = 0. \qquad (8)$$

The quadratic expression on the left-hand side cannot be factorised using integers, so the quadratic equation cannot be solved in this way. However, from Figure 27, which shows the graph of the equation $y = x^2 + 3x - 6$, you can see that this equation has two solutions, which are approximately $x = -4.4$ and $x = 1.4$.

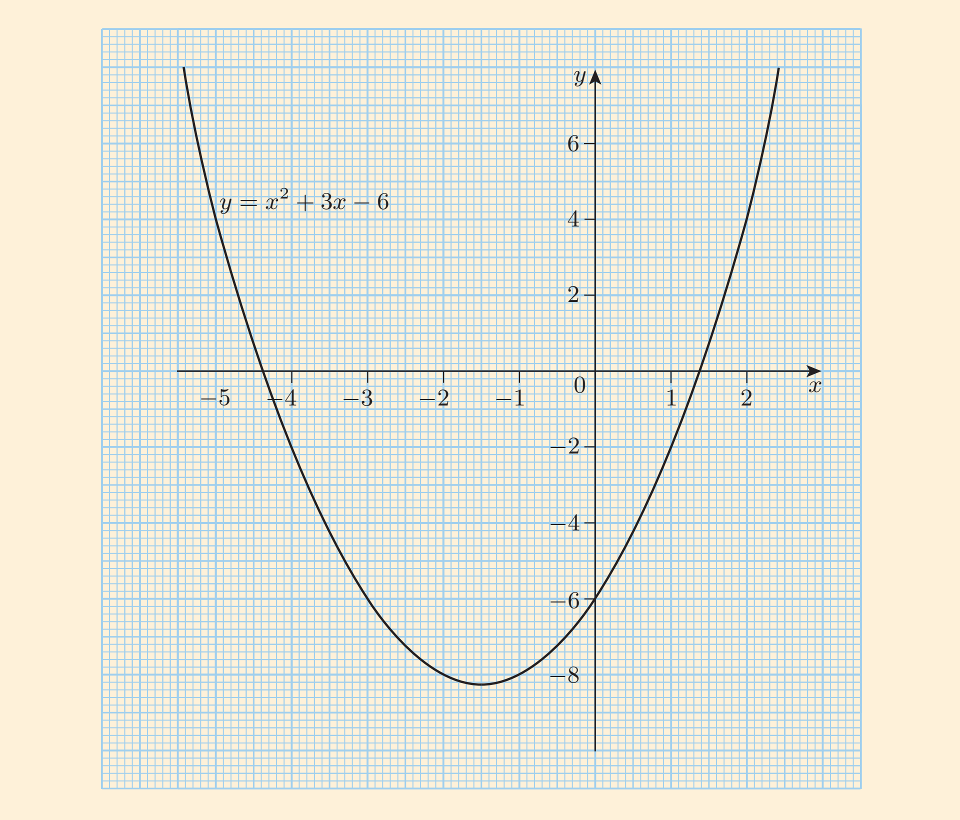


Figure 27 The graph of $y = x^2 + 3x - 6$

If you use Graphplotter to plot the graph of the quadratic function corresponding to a quadratic equation, then you can obtain the solutions more accurately.

For example, suppose that you want to find the solutions of equation (8) to two decimal places. To do this, you plot the graph of the equation $y = x^2 + 3x - 6$ on Graphplotter, and tick the 'Trace' option, which allows you to find the coordinates of points on the graph by moving the cursor.

Then you zoom in on a point where the graph crosses the x-axis, until the x-coordinates of the trace points are given to at least three decimal places (that is, at least one more decimal place than the precision that you eventually want). You can zoom in by clicking repeatedly on the 'Zoom in' button or using the mouse wheel – you will have to drag the graph or use the arrow buttons to keep the crossing point visible in the graph window. Alternatively, you can type appropriate values into the x min, x max, y min and y max boxes, to set the minimum and maximum values of the axis scales.

If you can find two trace points, one below the crossing point and one above, whose x-coordinates are the same when rounded to two decimal places, then the x-intercept must also be the same when rounded to two decimal places.

The next example demonstrates how to use this method to find each solution of equation (8).

Example 3 *Using Graphplotter to find approximate solutions*

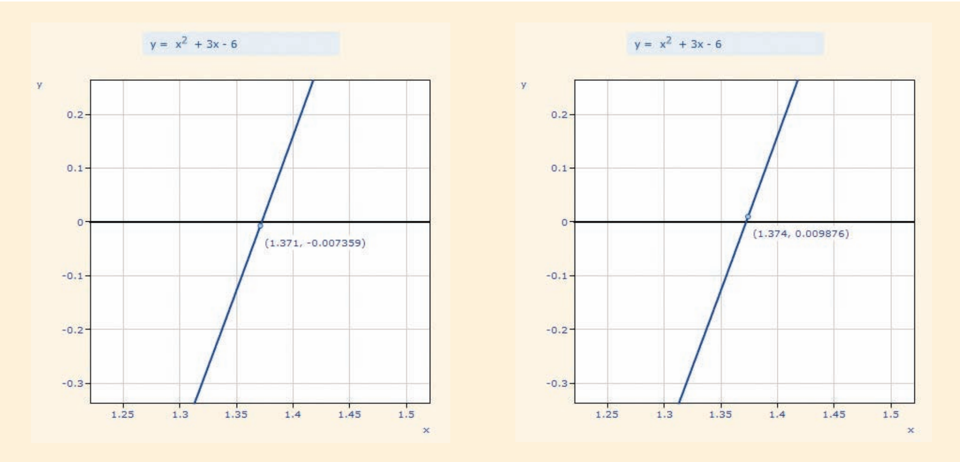
Use Graphplotter to find the solutions of the quadratic equation

$$x^2 + 3x - 6 = 0$$

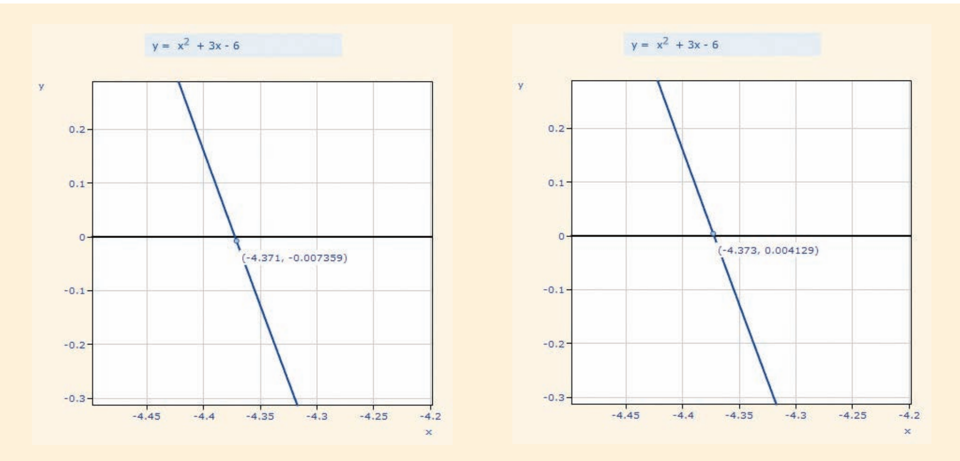
to two decimal places.

Solution

The two screenshots below are obtained by plotting the graph of $y = x^2 + 3x - 6$ and zooming in on one of the points where the graph crosses the x-axis. They show that there is a point on the graph below the x-axis with an x-coordinate of 1.371 to three decimal places, and a point on the graph above the x-axis with an x-coordinate of 1.374 to three decimal places. Since both of these values are 1.37 when rounded to two decimal places, and an x-intercept lies between them, it is also 1.37 to two decimal places. So one of the solutions of the equation is 1.37 (to 2 d.p.).



The two screenshots below are obtained by zooming in on the other crossing point. They show that there is a point on the graph below the x-axis with an x-coordinate of -4.371 to three decimal places, and a point on the graph above the x-axis with an x-coordinate of -4.373 to three decimal places. Since both of these values are -4.37 when rounded to two decimal places, and an x-intercept lies between them, it is also -4.37 to two decimal places. So the other solution of the equation is -4.37 (to 2 d.p.).



It is easier to use the Trace option on Graphplotter for the method demonstrated in Example 3 if the graph doesn't look too horizontal or too vertical at the crossing point, so sometimes it might be helpful to adjust the scale on the y-axis to achieve this.

Remember also that Graphplotter does not display *trailing zeros*, so, for example, an x-coordinate of 2.70 is displayed as 2.7.

In the next activity you can try the method of Example 3 for yourself.

Activity 12 *Solving a quadratic equation graphically*

Graphplotter

Use Graphplotter, with the 'One graph' tab selected. On the Options page, ensure that 'Trace' is ticked, and 'Coordinates' and 'y-intercept' are not ticked.

(a) Plot the graph of $y = -x^2 + 2x + 7$.

(b) Hence find the solutions of the quadratic equation $-x^2 + 2x + 7 = 0$, to two decimal places.

If you used Graphplotter to test Example 3 for yourself, then you may have reset the axis scales by zooming in, and so the graph in Activity 12 may not be visible. You can return the axis scales to their default values by pressing 'Autoscale'.

You can use a similar method to find the x-coordinate of any point on a graph, given its y-coordinate. Try this in the next activity.

Activity 13 *Using Graphplotter to find an x-coordinate*

Graphplotter

Use Graphplotter, with the 'One graph' tab selected. On the Options page, ensure that 'Trace' is ticked, and 'Coordinates' and 'y-intercept' are not ticked.

In Subsection 1.1, on page 131, you saw that if a ball is dropped out of a window at a height of 26 m, then its height y metres after it has been dropping for x seconds is given by the equation

$$y = 26 - 4.9x^2.$$

On page 131 the variables t and h were used instead of x and y, but x and y are used here because they are the variables used by Graphplotter.

Use Graphplotter to find the time taken by the ball to fall to a height of 20 m, to the nearest tenth of a second. To do this, plot the graph of the equation above, and set the x-axis scale to be 0 to 3 and the y-axis scale to be 0 to 30. Then move the cursor left and right to find a point on the graph with a y-coordinate just less than 20 and a point on the graph with a y-coordinate just greater than 20. You should find that you already have the precision that you need in the x-coordinates – there is no need to zoom in.

In Activity 13 you used Graphplotter to find the x-coordinate of a point on a graph, given its y-coordinate. Remember that it is straightforward to go the other way; that is, to find the y-coordinate of a point on a graph, given its x-coordinate. To do this, you first need to make sure that 'Trace' is ticked, then type the x-coordinate into the x-coordinate box, press 'Enter' and look at the coordinates of the trace point.

This method for finding y-coordinates was given in Activity 2 on page 131.

Graphplotter calculates y-coordinates from x-coordinates, so, for example, if a trace point on a graph is shown with coordinates $(1.87, 12.6052)$, then it means that an x-coordinate of exactly 1.87 gives a y-coordinate of 12.6052 correct to four decimal places. Remember though that

Graphplotter does not display trailing zeros, so, for example, if the x-coordinate 2.07 gives a y-coordinate of 9.1440 to four decimal places, then these coordinates are displayed as $(2.07, 9.144)$.

You saw earlier that the graph of a quadratic function has two, one or zero x-intercepts. Notice that this tells you that every quadratic equation has either two, one or zero solutions.

3.2 The quadratic formula

The first person to give a formula for solving quadratic equations was the Indian mathematician Brahmagupta, in 628. He described the formula in words, but it was essentially the same as the modern quadratic formula.

The **quadratic formula**, given below, provides a systematic way to find the exact solutions of any quadratic equation.

The quadratic formula

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

You will see why the formula works later in the unit, but for now you should concentrate on how to use it. Here is an example.

Example 4 *Using the quadratic formula*

Use the quadratic formula to solve the equation $3x^2 - 2x - 5 = 0$.

Solution

Check that the equation is in the form $ax^2 + bx + c = 0$, and find the values of a, b and c.

Here $a = 3$, $b = -2$ and $c = -5$. Substituting into the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-5)}}{2 \times 3} \\ &= \frac{2 \pm \sqrt{4 + 60}}{6} \\ &= \frac{2 \pm \sqrt{64}}{6} \\ &= \frac{2 \pm 8}{6} \\ &= \frac{2+8}{6} \quad \text{or} \quad \frac{2-8}{6} \\ &= \frac{10}{6} \quad \text{or} \quad \frac{-6}{6} \\ &= \tfrac{5}{3} \quad \text{or} \quad -1. \end{aligned}$$

So the solutions are $x = \frac{5}{3}$ and $x = -1$.

The quadratic equation in Example 4 could alternatively have been solved by using factorisation, as shown below:

$$3x^2 - 2x - 5 = 0$$
$$(3x - 5)(x + 1) = 0$$
$$3x - 5 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = \tfrac{5}{3} \quad \text{or} \quad x = -1.$$

This working is shorter and simpler than the working for the quadratic formula. So it is always worth checking whether a given quadratic equation can be factorised easily before you resort to using the quadratic formula. Factorising is often the quickest way to solve a quadratic equation, and the least likely to lead to mistakes.

The quadratic equation in Example 5 below cannot easily be factorised. In fact, its solutions turn out to be irrational, which confirms that it cannot be factorised using any rational numbers.

Example 5 *Using the quadratic formula again*

Tutorial clip

Use the quadratic formula to solve the equation $2x^2 + 4x - 7 = 0$.

Solution

Here $a = 2$, $b = 4$ and $c = -7$. Substituting into the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4 \times 2 \times (-7)}}{2 \times 2} \\ &= \frac{-4 \pm \sqrt{16 + 56}}{4} \\ &= \frac{-4 \pm \sqrt{72}}{4} \\ &= \frac{-4 \pm 6\sqrt{2}}{4} \\ &= -1 \pm \tfrac{3}{2}\sqrt{2}. \end{aligned}$$

Note that $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$.

So the solutions are

$$x = -1 + \tfrac{3}{2}\sqrt{2} \quad \text{and} \quad x = -1 - \tfrac{3}{2}\sqrt{2}.$$

The answers to Example 5 were left in surd form, so that they are exact. If they had been the answers to a problem in a real-life context, or were to be used to plot points on a graph, then it would be more sensible to give them as decimal approximations.

Notice also that the surds in Example 5 are expressed in their simplest form, by writing $\sqrt{72}$ as $6\sqrt{2}$. If you give the solutions to a quadratic equation as surds, then you should write them in their simplest form, using the methods that you learned in Unit 3. Sometimes you might find it helpful to use your calculator – the calculator recommended for the module can simplify surds.

Notice that in the last line of the working in Example 5, the expression

$$\frac{-4 \pm 6\sqrt{2}}{4} \tag{9}$$

was expanded to give

$$-1 \pm \tfrac{3}{2}\sqrt{2}.$$

An alternative way to simplify expression (9) is to cancel the common factor 2 in the numerator and denominator, to give

$$x = \frac{-2 \pm 3\sqrt{2}}{2},$$

and then state the solutions as

$$x = \frac{-2 + 3\sqrt{2}}{2} \quad \text{and} \quad x = \frac{-2 - 3\sqrt{2}}{2}.$$

Either of these ways of writing the solutions is just as simple, and just as acceptable, as the other.

You can practise using the quadratic formula in the next activity.

Activity 14 *Using the quadratic formula*

Use the quadratic formula to solve the following quadratic equations.

(a) $x^2 + 6x + 1 = 0$ (b) $3x^2 - 8x - 2 = 0$

Remember to check that your quadratic equation is in the form $ax^2 + bx + c = 0$ before you apply the quadratic formula! If it is not in this form, then you must rearrange it before you identify the values of a, b and c. For example, if you want to solve the equation

$$3x^2 + 2x = 4,$$

then you should first rearrange it as

$$3x^2 + 2x - 4 = 0,$$

which gives $a = 3$, $b = 2$ and $c = -4$.

Another thing to think about before you start to solve a quadratic equation is whether it is in its simplest form. You saw the following suggestions in Unit 9.

Simplifying a quadratic equation

- If the coefficient of x^2 is negative, then multiply the equation through by -1 to make this coefficient positive.
- If the coefficients have a common factor, then divide the equation through by this factor.
- If any of the coefficients are fractions, then multiply the equation through by a suitable number to clear them.

For example, you can simplify the quadratic equation

$$-2x^2 - 2x + 6 = 0$$

by multiplying through by -1 (to make the coefficient of x^2 positive) and

dividing through by 2 (to make all the coefficients smaller). This gives

$$x^2 + x - 3 = 0,$$

which you can then proceed to solve.

Activity 15 *Rearranging and solving a quadratic equation*

Clear the fraction in the quadratic equation

$$-x^2 = -x - \tfrac{3}{2}$$

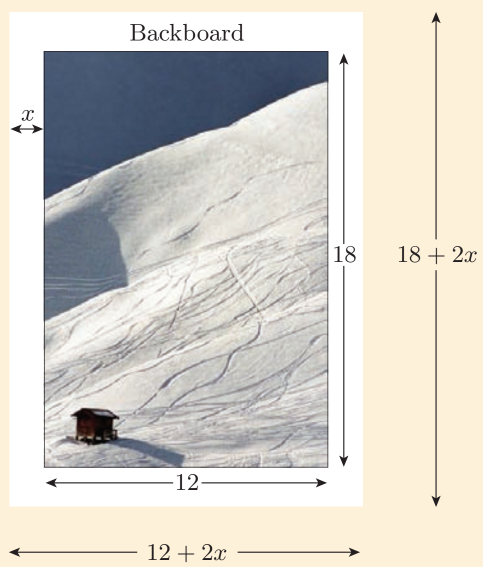
and use the quadratic formula to solve it.

Now that you have a way of solving quadratic equations that you could not solve by factorising, you can solve many more problems involving quadratic functions. Here is one for you to try.

Activity 16 *The picture framer's problem*

A picture framer always mounts photographs on a white rectangular backboard that has an area 50% larger than the area of the photograph, and whose dimensions are such that the white border around the photograph is the same width all the way round. The diagram below shows a mounted 12 inch by 18 inch photograph, with the width of the white border labelled as x. All the labelled lengths are in inches.

Standard photograph sizes in the UK are usually given in inches rather than centimetres. An inch is about 2.54 cm.



In this question you are asked to find the dimensions of the backboard, as follows.

(a) Explain in words why the width and height of the backboard, in inches, are $12 + 2x$ and $18 + 2x$, respectively.

(b) Use the fact that the area of the backboard is 50% larger than the area of the photograph to find the area of the backboard, in square inches.

(c) Find an algebraic expression for the area of the backboard in terms of x.

(d) Use your answers to parts (b) and (c) to show that $x^2 + 15x - 27 = 0$.

(e) Solve the quadratic equation in part (d), and hence find the width of the white border and the dimensions of the backboard to the nearest tenth of an inch.

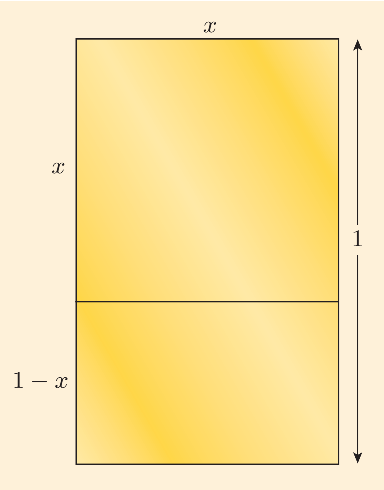


Figure 28 The golden rectangle

There is a story that the ancient Greeks believed that the most aesthetically pleasing shape of rectangle was the shape such that if you cut it into a square and a smaller rectangle, as shown in Figure 28, then the smaller rectangle has the same shape – that is, the same aspect ratio – as the original rectangle. This shape is known as the *golden rectangle*.

You can work out the aspect ratio of the golden rectangle as follows. If the length and width of the larger rectangle are 1 and x, respectively (in any units), then the larger and smaller rectangles have aspect ratios $1 : x$ and $x : (1 - x)$, respectively, as you can see from Figure 28. Since these two aspect ratios are equivalent, and the first aspect ratio is equivalent to $x : x^2$, the number x must satisfy the equation $x^2 = 1 - x$. You can use the quadratic formula to find the solutions of this equation. The solutions are $\frac{1}{2}(-1 \pm \sqrt{5})$, only one of which, $\frac{1}{2}(-1 + \sqrt{5})$, is positive. So the aspect ratio of the golden rectangle is $1 : \frac{1}{2}(-1 + \sqrt{5})$, and this ratio is known as the *golden ratio*.

Finally in this subsection, it is worth noticing how the quadratic formula relates to the equation of the axis of symmetry of a parabola. The quadratic formula can be rearranged slightly as

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

You can see from this form of the formula that the value of x halfway between the two solutions of the quadratic equation $ax^2 + bx + c = 0$ is $x = -b/(2a)$. This is the equation that you saw on page 147 for the axis of symmetry of the parabola with equation $y = ax^2 + bx + c$, as you would expect.

3.3 The number of solutions of a quadratic equation

You have seen that some quadratic equations have no solutions – this occurs when the corresponding graph has no x-intercepts. If you try to use the quadratic formula to solve such a quadratic equation, then the fact that there are no solutions quickly becomes clear. For example, consider the equation

$$x^2 + 7x + 13 = 0.$$

Here $a = 1$, $b = 7$ and $c = 13$, and substituting these values into the quadratic formula gives

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 13}}{2 \times 1} \\ &= \frac{-7 \pm \sqrt{49 - 52}}{2} \\ &= \frac{-7 \pm \sqrt{-3}}{2}. \end{aligned}$$

This expression involves the number $\sqrt{-3}$, but there is no such number, because negative numbers do not have square roots. So the equation has no solutions. This is confirmed by the graph in Figure 29.

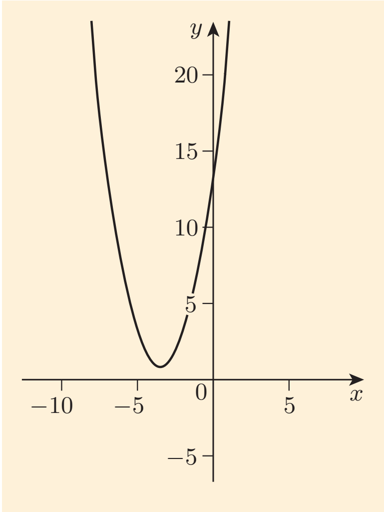


Figure 29 The graph of $y = x^2 + 7x + 13$

From this example you can see that it is the value of the expression $b^2 - 4ac$, which appears under the square root sign in the quadratic formula, that determines whether a quadratic equation has any solutions. In general, if $b^2 - 4ac$ is negative, then there are no solutions.

The value of $b^2 - 4ac$ also determines whether a quadratic equation has two solutions or just one. For example, consider the equation

$$4x^2 - 12x + 9 = 0.$$

Here $a = 4$, $b = -12$ and $c = 9$, and substituting these values into the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 4 \times 9}}{2 \times 4} \\ &= \frac{12 \pm \sqrt{144 - 144}}{8} \\ &= \frac{12 \pm \sqrt{0}}{8} \\ &= \tfrac{3}{2}. \end{aligned}$$

So this equation has only one solution. This is confirmed by the graph in Figure 30.

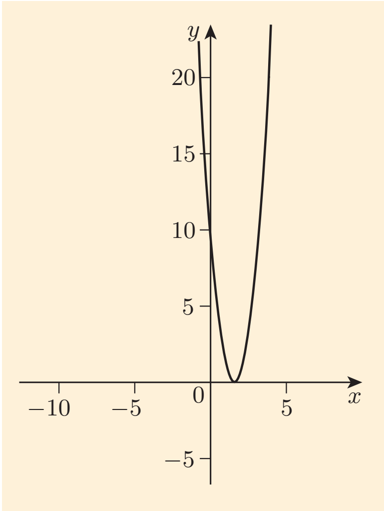


Figure 30 The graph of $y = 4x^2 - 12x + 9$

You can see that, in general, if $b^2 - 4ac$ is zero, then there is only one solution.

The value $b^2 - 4ac$ is called the **discriminant** of the quadratic expression $ax^2 + bx + c$.

The number of solutions of a quadratic equation

The quadratic equation $ax^2 + bx + c = 0$ has:

- two solutions if $b^2 - 4ac > 0$ (the discriminant is positive)
- one solution if $b^2 - 4ac = 0$ (the discriminant is zero)
- no solutions if $b^2 - 4ac < 0$ (the discriminant is negative).

Activity 17 *Predicting the number of solutions of a quadratic equation*

Use the discriminant to determine whether each of the following quadratic equations has two, one or no solutions. Find any solutions.

(a) $9x^2 + 30x + 25 = 0$ (b) $x^2 - 4x - 2 = 0$ (c) $-3x^2 + 5x = 4$

The first extensive discussion of complex numbers, including the formulation of how to add, subtract, multiply and divide them, was provided by the sixteenth-century Italian mathematician and engineer Rafael Bombelli in his book *Algebra* of 1572.

Although some quadratic equations have no solutions among the real numbers, all quadratic equations have either one or two solutions among the *complex numbers*, which were mentioned in Unit 3. The complex numbers consist of all the usual real numbers, together with many 'imaginary' numbers, including the square roots of negative numbers. Finding the imaginary solutions of a quadratic equation can be more useful than you might think! For example, they are used in many engineering mathematical models, such as those used to design car suspensions. You can learn about the imaginary solutions of quadratic equations if you go on to study more mathematics.

3.4 Vertically-launched projectiles

Imagine dropping a ball from the top of a cliff of height 12 m. From Section 1, you know that after t seconds the ball will have fallen $\frac{1}{2}gt^2$ metres, where g is the acceleration due to gravity (about $9.8\,\mathrm{m/s^2}$). So its height h metres above the ground below the cliff after t seconds is given by the equation

$$h = 12 - \tfrac{1}{2}gt^2.$$

Now imagine throwing the ball vertically up from the top edge of the cliff, with an initial speed of 2.8 m/s, instead of just dropping it. There are two factors influencing the motion of the ball – gravity and its initial speed. You can find a formula for the height of the ball after t seconds by considering the effects of both factors separately, and adding them together.

If there were no gravity (or any other forces acting on the ball), and you threw the ball upwards with a speed of 2.8 m/s, then it would continue to move at that constant speed, and so after t seconds its height would have increased by $2.8t$ metres.

On the other hand, if the ball had no initial speed, but just moved under the effect of gravity, then after t seconds its height would have decreased by $\frac{1}{2}gt^2$ metres.

Putting these two facts together, and using the fact that the initial height of the ball is 12 m, gives the following formula for the height h metres of the ball after t seconds:

$$h = 12 + 2.8t - \tfrac{1}{2}gt^2.$$

Writing the terms on the right-hand side in their usual order (with the term in t^2 first and the constant term last), and using the fact that $g = 9.8$, gives

$$h = -4.9t^2 + 2.8t + 12.$$

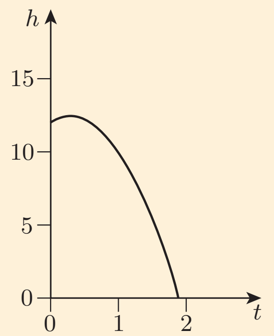


Figure 31 The graph of $h = -4.9t^2 + 2.8t + 12$

Figure 31 shows the graph of this equation. You can see that, as you would expect, the ball initially rises, and then starts to fall. It reaches the ground below the cliff in a little less than 2 seconds.

You can use the quadratic formula to find a more accurate estimate for the time that the ball takes to reach the ground. This is given by the value of t for which $h = 0$, so you need to solve the quadratic equation

$$-4.9t^2 + 2.8t + 12 = 0.$$

Before using the quadratic formula, let's multiply through by -1 to make the coefficient of t^2 positive. This is not essential, but it makes the numbers slightly easier to work with. This gives

$$4.9t^2 - 2.8t - 12 = 0.$$

Here $a = 4.9$, $b = -2.8$ and $c = -12$, and substituting these numbers into the quadratic formula gives

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2.8) \pm \sqrt{(-2.8)^2 - 4 \times 4.9 \times (-12)}}{2 \times 4.9} \\ &= \frac{2.8 \pm \sqrt{7.84 + 235.2}}{9.8} \\ &= \frac{2.8 \pm \sqrt{243.04}}{9.8}. \end{aligned}$$

Using a calculator gives

$$t = -1.3 \quad \text{or} \quad t = 1.9 \quad \text{(to 1 d.p.)}.$$

The negative solution does not make sense in this context, so we disregard it. So the ball reaches the ground below the cliff after approximately 1.9 seconds.

In general, suppose that an object is launched upwards from an initial height h_0 metres with an initial speed v_0 metres per second. Then after t seconds, the increase in height due to the initial speed is v_0t metres, and the decrease in height due to gravity is $\frac{1}{2}gt^2$ metres. This gives the general formula below.

Think of h_0 and v_0 as single symbols; they represent constants. They are pronounced as *h-nought* and *v-nought*, respectively. The significance of the zeros is that the symbols denote the height and speed of the object at time $t = 0$. The symbol v is often used for speed, as it is the first letter of 'velocity'.

The motion of a vertically-launched projectile

If an object is launched upwards from an initial height h_0 with an initial speed v_0, then after time t its height h is given by

$$h = -\tfrac{1}{2}gt^2 + v_0t + h_0, \qquad (10)$$

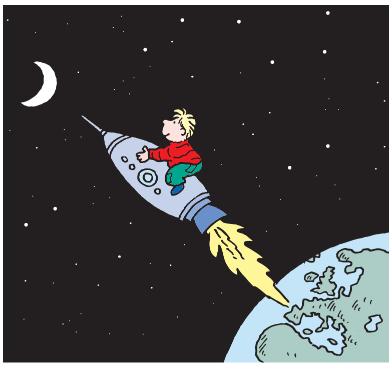
where g is the acceleration due to gravity, which is about $9.8\,\text{m/s}^2$.

Of course, as with all the formulas for falling objects in this unit, this formula is only a model, and in real life the motion of an object will differ to some extent from that predicted by the formula, largely due to air resistance.

Try using this formula in the activity below.

Activity 18 *Finding the descent time of a toy rocket*

The fuel in a toy rocket runs out at a height of 155 m above the ground, at which point it has an upwards speed of 49 m/s. After how many seconds does the toy rocket return to the ground? Give your answer to one decimal place.



Formula (10) in the box above can be used to predict the motion of a projectile launched at an angle, such as a cannonball fired at an angle of 45° to the ground, as shown in Figure 17 on page 136. This is done by considering the horizontal and vertical motion of the projectile separately,

in the way that you saw in Section 1. Formula (10) models the vertical motion, and the horizontal motion is modelled by the usual distance–speed–time formula for constant speed, as before. Before these formulas can be applied, it is necessary to calculate how much of the initial speed contributes to vertical motion and how much contributes to horizontal motion. You can find out how to do this in higher-level mathematics modules – it involves trigonometry, which you will learn about in Unit 12.

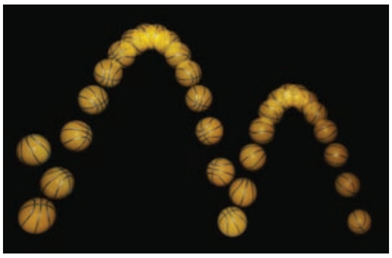


Figure 32 The parabolas formed by the trajectories of a bouncing ball

As mentioned in Section 1, the trajectory of such a projectile is always parabolic, which is why, for example, the shape of a jet of water in a fountain is a parabola. Figure 32 shows the parabolic trajectories of a bouncing ball.

In this section you have learned two new ways to solve quadratic equations: by using a graph, which gives approximate solutions, and by using the quadratic formula, which gives exact solutions. You have applied these methods to some practical problems.

4 Completing the square

In this section you will learn a useful way of rearranging a quadratic expression, which is called *completing the square*. This method allows you to understand why the graph of any quadratic function is the same shape as the graph of an equation of the form $y = ax^2$, but shifted horizontally or vertically or both.

The method also gives you an alternative way to find the vertex of a parabola, and an alternative way to solve a quadratic equation. Once you have learned the method, it can be a quick way of doing these things.

Completing the square is also the idea behind the quadratic formula – it is why the quadratic formula works, as you will see.

4.1 Shifting parabolas

We begin by looking at what happens when you start with a parabola of the form $y = ax^2$ and shift it relative to the axes.

Figure 33 shows the graph of the equation $y = 2x^2$. As you know, its vertex is the origin. A second parabola has been drawn on the same axes – this parabola is exactly the same shape as the first parabola, but it is shifted three units to the right and one unit up.

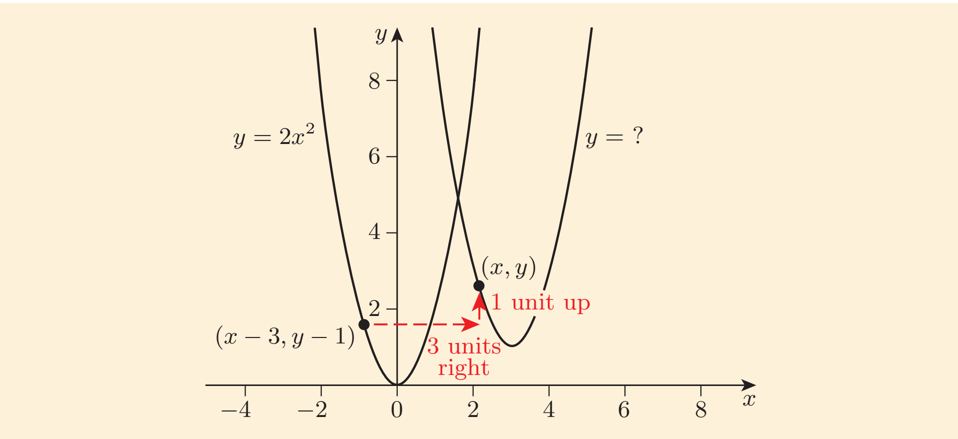


Figure 33 The graph of $y = 2x^2$ and the graph obtained by shifting it 3 units to the right and 1 unit up

Let's consider whether we can use the equation of the first parabola to work out an equation for the second. Let (x, y) be any point on the *second* parabola. Then (x, y) is the result of shifting a corresponding point on the first parabola, as shown in the diagram. This corresponding point is 3 units left and 1 unit down from (x, y), so its coordinates are

$$(x - 3, y - 1).$$

Because *this* point lies on the first parabola, *its* coordinates satisfy the equation

$$y\text{-coordinate} = 2 \times x\text{-coordinate}^2.$$

So

$$y - 1 = 2(x - 3)^2.$$

Since x and y refer to the second parabola, this is an equation that is satisfied by every point (x, y) on the second parabola, so it is the equation of the second parabola. It can be rearranged to get y by itself on the left-hand side, as follows:

$$y = 2(x - 3)^2 + 1.$$

The equation can be left in this form, or it can be multiplied out to give the more usual form:

$$\begin{aligned} y &= 2(x^2 - 6x + 9) + 1 \\ &= 2x^2 - 12x + 18 + 1 \\ &= 2x^2 - 12x + 19. \end{aligned}$$

In general, suppose that the parabola with equation $y = ax^2$ is shifted right by h units and up by k units. The numbers h and k can be positive, negative or zero, so the actual shift could be to the right or left, or neither, and up or down, or neither. Then each point (x, y) on the second parabola is a shift of the point $(x - h, y - k)$ on the first parabola, as illustrated in Figure 34.

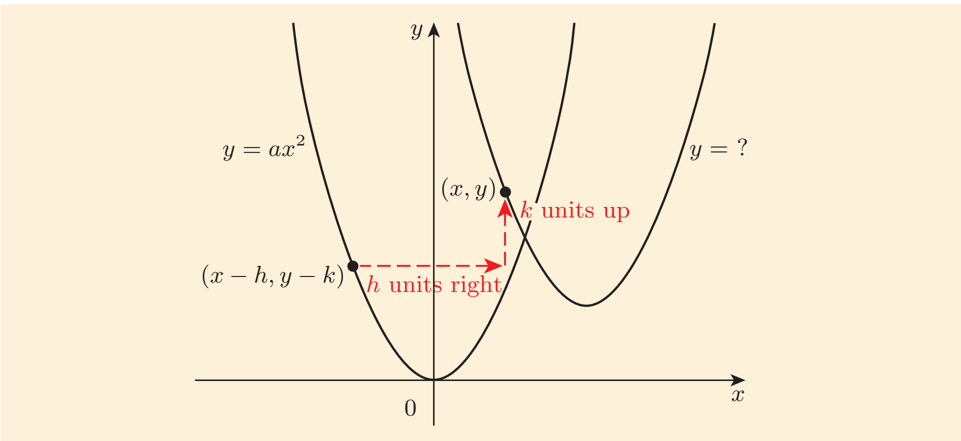


Figure 34 The graph of $y = ax^2$ and the graph obtained by shifting it h units right and k units up

By the same argument as before, we have

$$y - k = a(x - h)^2,$$

or equivalently,

$$y = a(x - h)^2 + k, \qquad (11)$$

and this is the equation of the second parabola.

Any equation of form (11) can be multiplied out to give the usual form of the equation of a parabola, $y = ax^2 + bx + c$.

However, the really crucial fact is that you can always go in the other direction, too. That is, *any* equation of the form $y = ax^2 + bx + c$, where a, b and c are constants with $a \neq 0$, can be rearranged into form (11) (where the constants h and k can be positive, negative or zero). You'll learn how to do this in this section.

The fact that the equation of *every* quadratic function is equivalent to an equation of form (11) explains why the graph of every quadratic function is a shift of a parabola of the form $y = ax^2$. You saw that this *seemed* to be the case when you used Graphplotter to investigate the shapes of parabolas in Section 2.

Since h and k can be positive, negative or zero, the fact stated above can be restated as below.

Completed-square form

Every expression of the form $ax^2 + bx + c$, where $a \neq 0$, can be rearranged into the form

$$a\left(x + \text{a number}\right)^2 + \text{a number},$$

where each of the two numbers in this expression can be positive, negative or zero.

This is called the **completed-square form**.

The process of finding the completed-square form of a quadratic expression is called **completing the square**.

It was mentioned earlier that one reason for completing the square is that it gives you a quick way to find the vertex of a parabola. To see how to do this, suppose that you have rearranged the equation of a parabola into the form of equation (11); that is,

$$y = a(x - h)^2 + k.$$

This tells you that the parabola has the same shape as the parabola with equation $y = ax^2$, but shifted h units to the right and k units up, as shown in Figure 34 on the previous page. Since the vertex of the parabola with equation $y = ax^2$ is $(0, 0)$, the vertex of the shifted parabola is (h, k).

This useful result is summarised below.

The parabola with equation

$$y = a(x - h)^2 + k$$

has vertex (h, k).

For example, suppose that you have rearranged the equation of a parabola into the completed-square form

$$y = 3(x + 5)^2 + 8.$$

Here $h = -5$ and $k = 8$, so the vertex is $(-5, 8)$.

It can be difficult to remember exactly how to obtain the coordinates of the vertex from the completed-square form. One way to remember it is to use the fact that the vertex always corresponds to the minimum or maximum value of y (depending on whether the parabola is u-shaped or n-shaped). For example, consider again the completed-square form

$$y = 3(x+5)^2 + 8.$$

The parabola that is the graph of this equation is u-shaped, because when you multiply out the brackets, the coefficient of x^2 will be 3, which is positive. So the vertex of this parabola corresponds to the *minimum* value of y.

Notice that the equation contains the expression $(x+5)^2$, and the minimum value of this expression is 0, because the square of a number is never negative. So the minimum value of $3(x+5)^2$ is also 0, and hence the minimum value of the whole expression $3(x+5)^2 + 8$ is 8. That is, the y-coordinate of the vertex is 8.

This minimum value occurs when the expression that is squared is zero, that is, when $x+5=0$ or $x=-5$. So the x-coordinate of the vertex is -5, and hence the vertex is $(-5, 8)$, as found above.

Here is another example of finding the vertex from a completed-square form in this way.

Example 6 *Finding the vertex of a parabola from its completed-square form*

State whether the parabola with equation

$$y = -(x-2)^2 - 3$$

is u-shaped or n-shaped, and write down the coordinates of its vertex.

Solution

The parabola is n-shaped, because the coefficient of x^2 is -1, which is negative.

The minimum value of $(x-2)^2$ is 0, so the *maximum* value of $-(x-2)^2$ is 0, and hence the maximum value of $-(x-2)^2 - 3$ is -3.

This occurs when $x-2=0$, that is, when $x=2$.

The vertex is $(2, -3)$.

Here are some similar examples for you to try.

Activity 19 *Finding the vertices of parabolas from completed-square forms*

For each of the following equations, state whether the parabola is u-shaped or n-shaped, and write down the coordinates of its vertex.

(a) $y = (x+1)^2 + 5$ (b) $y = -2(x+3)^2 + 7$ (c) $y = 7(x-1)^2 - 4$

(d) $y = -\left(x + \frac{1}{2}\right)^2 - 1$ (e) $y = x^2 + 3$ (f) $y = (x-2)^2$

In the next subsection you will learn the basic method for completing the square in a quadratic expression.

4.2 Completing the square in quadratics of the form $x^2 + bx + c$

In this subsection you will learn how to complete the square in quadratic expressions in which the coefficient of x^2 is 1. Other quadratics are covered in Subsection 4.4.

The completed-square form of a quadratic in which the coefficient of x^2 is 1 is

$$\left(x + \text{a number}\right)^2 + \text{a number}\,.$$

This is the expression given in the first pink box on page 162, with $a = 1$.

We begin by looking at completing the square in quadratics in which not only does x^2 have coefficient 1, but the constant term is zero – that is, expressions such as $x^2 + 8x$, $x^2 + 10x$ or $x^2 - 6x$. In other words, we will look at quadratics of the form $x^2 + bx$.

Completing the square in quadratics of the form $x^2 + bx$

To see how to complete the square in a quadratic expression of this form, first consider the following examples of expanding squared brackets.

You saw how to expand squared brackets in Unit 9.

$$(x + 1)^2 = x^2 + 2x + 1$$
$$(x - 2)^2 = x^2 - 4x + 4$$
$$(x + 3)^2 = x^2 + 6x + 9$$

In general, for any number p, positive, negative or zero,

$$(x + p)^2 = x^2 + 2px + p^2.$$

The expression on the right-hand side of each equation above is of the form $x^2 + bx$ plus an extra number. If you subtract this extra number from both sides of each equation, and swap the sides, then you obtain

$$x^2 + 2x = (x + 1)^2 - 1,$$
$$x^2 - 4x = (x - 2)^2 - 4,$$
$$x^2 + 6x = (x + 3)^2 - 9,$$

and in general,

$$x^2 + 2px = (x + p)^2 - p^2.$$

The expressions on the right-hand sides of the equations above are in completed-square form, so they are the completed-square forms of the expressions on the left. In each case the constant term in the brackets on the right-hand side is half of the coefficient of the term in x on the left. For example:

$$x^2 + 6x = (x + 3)^2 - 9$$

Half of the coefficient of x

Also, in each case the number that is subtracted on the right-hand side is the square of the constant term in the brackets.

For example:

$$x^2 + 6x = (x + 3)^2 - 9$$

The square of the number in brackets

You can see why this is by considering the general case:

$$x^2 + 2px = (x + p)^2 - p^2$$

Half of the coefficient of x

The square of the number in brackets

You can now see how to write down the completed-square form of any quadratic expression of the form $x^2 + bx$. First you write down $(x \sqcup)^2$, filling the gap with the number that is half of b, the coefficient of x. This ensures that you have a squared bracket that, when expanded, gives the terms $x^2 + bx$. However, it also gives an extra term, which is the square of the number in the gap. So you need to subtract this term to obtain a final completed-square form that is equivalent to $x^2 + bx$.

Here is an example.

Example 7 *Completing the square in a quadratic of the form $x^2 + bx$*

Write the quadratic expression $x^2 - 10x$ in completed-square form.

Solution

$$x^2 - 10x = (x - 5)^2 - 25$$

Halve this coefficient and write it here

Square the constant term in brackets and subtract it

You can use the general equation

$$x^2 + 2px = (x + p)^2 - p^2$$

as a 'formula for completing the square', if you prefer to think of the method this way.

Once you have found a completed-square form, you can check that it is correct by multiplying it out. For example, multiplying out the expression on the right-hand side of the equation in the solution to Example 7 gives

$$\begin{aligned}(x - 5)^2 - 25 &= x^2 - 10x + 25 - 25\\ &= x^2 - 10x,\end{aligned}$$

which is the same as the left-hand side, as expected.

Here are some examples of completing the square for you to try.

Activity 20 *Completing the square in quadratics of the form $x^2 + bx$*

Write the following quadratic expressions in completed-square form, and check your answers by multiplying out.

(a) $x^2 + 16x$ (b) $x^2 - 12x$ (c) $t^2 - 2t$ (d) $x^2 + 3x$

Completing the square in quadratics of the form $x^2 + bx + c$

To complete the square in a quadratic of the form $x^2 + bx + c$ whose constant term c is not zero, you just concentrate on the terms in x^2 and x, and complete the square for these terms in the same way as before. Then you have to collect the constant terms. This is illustrated in the example below.

Tutorial clip

Example 8 *Completing the square in quadratics of the form $x^2 + bx + c$*

Write the following quadratic expressions in completed-square form.

(a) $x^2 + 8x + 10$ (b) $x^2 - 3x + 5$

Solution

(a) First complete the square for the sub-expression $x^2 + 8x$, leaving the $+10$ unchanged.

$$x^2 + 8x + 10 = (x+4)^2 - 16 + 10$$

Then collect the constant terms.

$$= (x+4)^2 - 6$$

(Check:

$$\begin{aligned}(x+4)^2 - 6 &= x^2 + 8x + 16 - 6 \\ &= x^2 + 8x + 10.)\end{aligned}$$

(b) First complete the square for the sub-expression $x^2 - 3x$.

$$x^2 - 3x + 5 = \left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + 5$$

Then collect the constant terms.

$$\begin{aligned}&= \left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + \tfrac{20}{4} \\ &= \left(x - \tfrac{3}{2}\right)^2 + \tfrac{11}{4}.\end{aligned}$$

(Check:

$$\begin{aligned}\left(x - \tfrac{3}{2}\right)^2 + \tfrac{11}{4} &= x^2 - 3x + \tfrac{9}{4} + \tfrac{11}{4} \\ &= x^2 - 3x + 5.)\end{aligned}$$

Activity 21 *Completing the square in quadratics of the form $x^2 + bx + c$*

Write the following quadratic expressions in completed-square form, and check your answers by multiplying out.

(a) $x^2 + 6x - 3$ (b) $x^2 - 4x + 9$ (c) $p^2 - 12p - 5$

(d) $x^2 + x + 1$

Here is a summary of the method that you have seen in this subsection.

Strategy *To complete the square in a quadratic of the form* $x^2 + bx + c$

1. Rewrite the expression with the $x^2 + bx$ part changed to

   $$(x + p)^2 - p^2,$$

   where the number p is half of b.
2. Collect the constant terms.

Remember that in this strategy the number b, and hence the number p, can be either positive or negative. (If b is zero, then the quadratic is already in completed-square form.)

Later in the section you will see how to complete the square in quadratics in which the coefficient of x^2 is not 1. Before that, the next subsection tells you how to solve a quadratic equation by completing the square.

4.3 Solving quadratic equations by completing the square

As mentioned earlier, completing the square gives you another method of solving quadratic equations (and hence of finding the x-intercepts of parabolas).

To solve a quadratic equation in this way, you only ever need to complete the square in expressions of the form $x^2 + bx + c$, that is, in quadratics whose term in x^2 has coefficient 1. This is because you can always divide a quadratic equation through by the coefficient of x^2 to give a quadratic equation whose term in x^2 has coefficient 1. For example, if the quadratic equation is

$$5x^2 - 3x + 10 = 0,$$

then you can divide through by 5 to give

$$x^2 - \tfrac{3}{5}x + 2 = 0.$$

Dividing through by the coefficient of x^2 can turn some of the coefficients in the equation from whole numbers into fractions, but that doesn't matter, as fractions are treated in the same way as any other number when completing the square.

Once you have completed the square in the equation, you can solve the equation as follows. First rearrange it so that the square term and the constant term are on different sides, then take the square root of both sides and finally rearrange the equation again to obtain x by itself on one side. This is illustrated in the example below.

Example 9 *Solving a quadratic equation by completing the square*

Solve the quadratic equation $4x^2 + 8x - 1 = 0$.

Solution

$$4x^2 + 8x - 1 = 0$$

Divide through by the coefficient of x^2.

$$x^2 + 2x - \tfrac{1}{4} = 0$$

The Babylonians in about 1850–1650 BC were able to solve problems equivalent to quadratic equations. Their method was essentially one of completing the square, but they found only positive solutions, as their problems involved positive quantities such as length.

Complete the square.

$$(x+1)^2 - 1 - \tfrac{1}{4} = 0$$
$$(x+1)^2 - \tfrac{5}{4} = 0$$

Get the constant term on the right.

$$(x+1)^2 = \tfrac{5}{4}$$

Take the square root of both sides.

$$x + 1 = \pm\sqrt{\tfrac{5}{4}}$$
$$x + 1 = \pm\tfrac{1}{2}\sqrt{5}$$

Finally, get x by itself on the left.

$$x = -1 \pm \tfrac{1}{2}\sqrt{5}$$

You could write $x = \pm\frac{1}{2}\sqrt{5} - 1$ at this stage, but in an expression like this it is traditional to put the root last.

The solutions are $x = -1 + \frac{1}{2}\sqrt{5}$ and $x = -1 - \frac{1}{2}\sqrt{5}$.

Activity 22 *Solving quadratic equations by completing the square*

Solve the following quadratic equations by completing the square.

(a) $x^2 + 6x - 5 = 0$ (b) $2x^2 - 12x - 5 = 0$

The derivation of the quadratic formula

You can use the technique of completing the square to rearrange the general quadratic equation

$$ax^2 + bx + c = 0,$$

to obtain x by itself on the left-hand side. The method is the same as in Example 9, but a, b and c are not replaced by particular numbers – they just stay as they are throughout the manipulation. When you do this rearrangement, you end up with the quadratic formula.

The manipulation is given below – read it through if you would like to know why the quadratic formula works.

The equation is

$$ax^2 + bx + c = 0,$$

where $a \neq 0$.

The first step is to divide through by the coefficient of x^2:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Then you complete the square:

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

Next you get the constant terms on the right, and combine them into a single fraction:

$$\begin{aligned}\left(x+\frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2}-\frac{c}{a}\\ &= \frac{b^2}{4a^2}-\frac{4ac}{4a^2}\\ &= \frac{b^2-4ac}{4a^2}.\end{aligned}$$

Now you can take the square root of both sides:

$$\begin{aligned}x+\frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}}\\ &= \pm\frac{\sqrt{b^2-4ac}}{2a}.\end{aligned}$$

The last step is to get x by itself on the left-hand side:

$$\begin{aligned}x &= -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\\ &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\end{aligned}$$

This is the quadratic formula!

4.4 Completing the square in quadratics of the form $ax^2 + bx + c$

So far you have seen how to complete the square in quadratic expressions in which the coefficient of x^2 is 1. In this subsection you will see how to complete the square in quadratic expressions in which the coefficient of x^2 is *not* 1. Although this is not needed to solve quadratic equations, it is useful for finding the vertices of parabolas.

When you are faced with a problem that is different from those that you have seen before, a useful strategy is to try to change it into a form that you recognise. If you have to complete the square in a quadratic expression in which the coefficient of x^2 is not 1, then you can turn this problem into a problem of completing the square when the coefficient *is* 1, by taking the coefficient of x^2 out as a factor. You don't need to take the factor out of all the terms, but just the terms in x^2 and x. For example, to complete the square in the quadratic expression

$$2x^2 + 8x - 7,$$

you can take the common factor 2 out of the first two terms to obtain

$$2(x^2 + 4x) - 7.$$

Then you can complete the square in the quadratic *inside the brackets* in the way that you have seen. To obtain the final completed-square form for the whole expression you just need to simplify the results.

You can take the coefficient of x^2 out of the terms in x^2 and x in a quadratic expression even if it isn't a *common* factor of these terms. You'll see this shortly, but first the example below illustrates the method when the coefficient is a common factor.

Tutorial clip

Example 10 *Completing the square in quadratics of the form* $ax^2 + bx + c$

Write the following quadratic expressions in completed-square form.

(a) $2x^2 + 8x - 7$ (b) $-x^2 + 8x - 7$

Solution

(a) Concentrate on the sub-expression $2x^2 + 8x$. First take the coefficient of x^2 out of the sub-expression as a common factor.

$$2x^2 + 8x - 7 = 2\left(x^2 + 4x\right) - 7$$

Now the brackets contain a quadratic in which the coefficient of x^2 is 1. Complete the square in it in the usual way, keeping it enclosed within its brackets.

$$= 2\left((x+2)^2 - 4\right) - 7$$

Multiply out the *outer* brackets. Don't multiply out the inner brackets, because you want the square $(x+2)^2$ to appear in the final expression.

$$= 2(x+2)^2 - 8 - 7$$

Collect the constant terms.

$$= 2(x+2)^2 - 15$$

(Check:

$$\begin{aligned} 2(x+2)^2 - 15 &= 2(x^2 + 4x + 4) - 15 \\ &= 2x^2 + 8x + 8 - 15 \\ &= 2x^2 + 8x - 7.) \end{aligned}$$

(b) Concentrate on the sub-expression $-x^2 + 8x$. First take the minus sign out of the sub-expression.

$$-x^2 + 8x - 7 = -(x^2 - 8x) - 7$$

Complete the square in the quadratic inside the brackets.

$$= -((x-4)^2 - 16) - 7$$

Multiply out the *outer* brackets.

$$= -(x-4)^2 + 16 - 7$$

Collect the constant terms.

$$= -(x-4)^2 + 9$$

(Check:

$$\begin{aligned} -(x-4)^2 + 9 &= -(x^2 - 8x + 16) + 9 \\ &= -x^2 + 8x - 16 + 9 \\ &= -x^2 + 8x - 7.) \end{aligned}$$

In Example 10(a) the coefficient of x^2 was a common factor of the sub-expression consisting of the terms in x^2 and x, but of course this is not always the case. For example, consider the quadratic expression

$$2x^2 + 5x + 1.$$

To complete the square in a quadratic like this, you begin by taking out the coefficient of x^2 from the sub-expression just as if it *were* a common factor – this will create fractions. For the quadratic here, you obtain

$$2\left(x^2 + \tfrac{5}{2}x\right) + 1.$$

You can then go on to complete the square using the method demonstrated in Example 10. The final answer is

$$2\left(x + \tfrac{5}{4}\right)^2 - \tfrac{17}{8}.$$

Activity 23 *Completing the square in quadratics of the form* $ax^2 + bx + c$

Write the quadratic expressions below in completed-square form, and check your answers by multiplying out.

Use the completed-square forms to write down the vertices of the corresponding parabolas.

(a) $2x^2 - 4x - 1$ (b) $-x^2 - 8x - 18$

Here is a summary of the method that you have seen in this subsection.

Strategy *To complete the square in a quadratic of the form* $ax^2 + bx + c$

1. Rewrite the expression with the coefficient a of x^2 taken out of the $ax^2 + bx$ part as a factor. This generates a pair of brackets.
2. Complete the square in the simple quadratic inside the brackets, remembering to keep it enclosed within its brackets. This generates a second pair of brackets, inside the first pair.
3. Multiply out the *outer* brackets.
4. Collect the constant terms.

Writing the equation of a parabola in completed-square form gives you another way to find some of the information that you need to sketch it. You can read off the coordinates of the vertex immediately, and you can use the completed-square form to solve the quadratic equation that gives the x-intercepts.

In this section you have seen how to complete the square in any quadratic expression. You have seen that this method explains why the graphs of all quadratic functions are shifts of the graphs of equations of the form $y = ax^2$, and that it also explains why the quadratic formula works. You have also seen how to use the method to solve quadratic equations and to find vertices of parabolas.

5 Maximisation problems

In some real-life situations it is useful to find the maximum possible value of some quantity and to find the circumstances under which that maximum value is obtained. For example, a shopkeeper may want to know what his prices should be if he is to make the maximum possible profit. The problem of finding the maximum value of a quantity and the circumstances under which it is obtained is known as a **maximisation problem**.

If the situation can be modelled with a quadratic function, then the problem is solved by finding the vertex of its graph. You will see some examples of this in this section.

You have seen several methods for finding the vertex of the graph of a quadratic function in this unit. They are summarised in the box below.

To find the vertex of a parabola from its equation

To find the vertex of the parabola with equation $y = ax^2 + bx + c$, use any of the following methods.

- Use the formula $x = -b/(2a)$ to find the x-coordinate, then substitute into the equation of the parabola to find the y-coordinate.
- Find the x-intercepts (if there are any); then the value halfway between them is the x-coordinate of the vertex. Find the y-coordinate by substituting into the equation of the parabola.
- Complete the square: the parabola with equation $y = a(x - h)^2 + k$ has vertex (h, k).
- Plot the parabola using Graphplotter and read off the approximate coordinates of the vertex.

The fourth method of finding the vertex, plotting the parabola using Graphplotter, is useful if you need only approximate answers. It can also be a helpful way to check an answer that you have found using one of the other methods.

Often the trickiest part of solving a maximisation problem is finding a quadratic function that models the situation, and you will see some ideas for doing that in this section too.

5.1 The maximum height of a vertically-launched projectile

You have already seen how to model the motion of a vertically-launched projectile with a quadratic function. You saw in Subsection 3.4 that if a projectile is launched vertically upwards from an initial height h_0 with an initial speed v_0, then its height h after time t is given by the equation

$$h = -\tfrac{1}{2}gt^2 + v_0 t + h_0, \qquad (12)$$

where g is the acceleration due to gravity (about $9.8\,\mathrm{m/s^2}$).

In the example below this model is used to find the maximum height reached by a distress flare fired vertically.

Example 11 *Finding the height reached by a distress flare*

A distress flare is fired vertically upwards from a height of 2 m at an initial speed of 52 m/s. Find the height that it reaches, and the time that it takes to reach this height, to two significant figures.

Solution

The motion of the flare is modelled by equation (12) with $h_0 = 2$, $v_0 = 52$ and $g = 9.8$. So its height h in metres after t seconds is given by

$$h = -4.9t^2 + 52t + 2.$$

The maximum value of h is at the vertex of this parabola. The t-coordinate of the vertex is given by

$$t = -\frac{b}{2a},$$

where $a = -4.9$ and $b = 52$, so it is

$$t = -\frac{52}{2 \times (-4.9)} = 5.3061\ldots = 5.3 \text{ (to 2 s.f.)}.$$

Substituting $t = 5.3061\ldots$ into the equation of the parabola gives

$$\begin{aligned} h &= -4.9t^2 + 52t + 2 \\ &= -4.9 \times (5.3061\ldots)^2 + 52 \times (5.3061\ldots) + 2 \\ &= 140 \text{ (to 2 s.f.)}. \end{aligned}$$

So the vertex is approximately $(5.3, 140)$.

State a conclusion in the context of the question.

Hence the flare reaches a height of about 140 metres, and it takes about 5.3 seconds to reach this height.

As mentioned earlier, when you solve a problem like the one in Example 11, it is useful to check your answers using Graphplotter. Figure 35 shows a Graphplotter graph of the quadratic function in Example 11, and you can see that the vertex seems to be about $(5.3, 140)$, as expected.

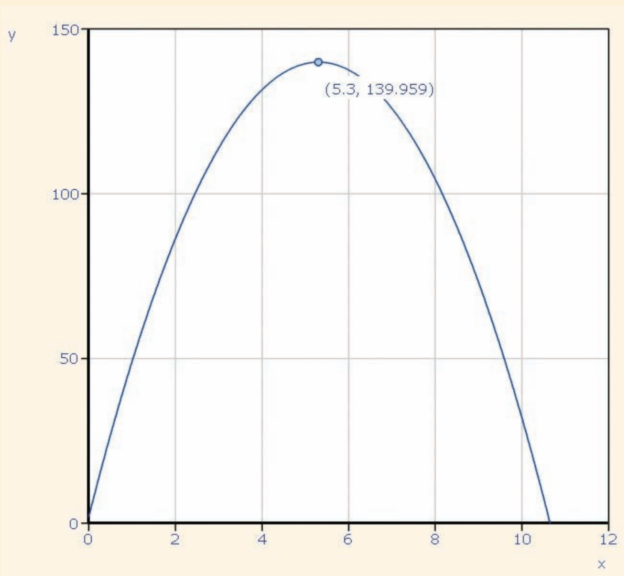


Figure 35 The graph of $y = -4.9x^2 + 52x + 2$

Here is a similar example for you to try.

Activity 24 *Finding the height reached by a ball thrown upwards*

A ball is thrown vertically upwards from a height of 1.6 m with an initial speed of 15 m/s. Find the maximum height reached by the ball, to the nearest metre.

5.2 Maximising yields

Figure 36 Revenue against possible price increase

Finding the maximum value of a quadratic function can help with decisions about how to maximise the yield from a business.

For example, the supplier of a saleable item or service can often increase the money that he makes – his *revenue* – by increasing his price. This may cause a drop in demand, but if the price increase is not too much, then the extra revenue from the increased price may outweigh the loss in revenue from the drop in demand. If the price increase is too much, however, then a large drop in demand can cause the overall revenue to decrease.

So as the possible price increase gets larger, the resulting revenue tends to increase and then decrease again, as illustrated in Figure 36. The optimum price increase can often be found by modelling the situation with a quadratic function, and finding the vertex. Here is an example to illustrate these ideas.

Example 12 *Maximising a boatman's revenue*

A boatman has 30 boats for hire. He finds that if he charges £20 an hour, then all his boats are hired, but generally for each £1 increase in the hire charge, one boat fewer is hired. Determine the amount by which the boatman should increase his price if he wants to maximise his revenue.

Solution

Find a quadratic function to model the situation. The first step is to decide what the variables should be.

The quantity that is to be maximised is the revenue, so denote that by £r.

The second variable could denote the price, rather than the price increase, but it is slightly easier to work with the increase.

The quantity that makes the revenue increase and decrease is the price increase, so denote that by £i.

Now find a formula for r in terms of i. You might be able to write it down straight away. If not, try looking at a few numerical examples, like those in the table below.

Price increase (£)	Hire price (£)	Number of boats hired	Revenue (£)
0	20	30	20×30
1	21	29	21×29
2	22	28	22×28
$\vdots$	$\vdots$	$\vdots$	$\vdots$

In general, if the price increase in £ is i, then the hire price in £ is $20 + i$,

and the number of boats hired is $30 - i$. So the overall revenue per hour in £ is given by

$$r = (20 + i)(30 - i).$$

This is the equation of a parabola, with the right-hand side in factorised form. To find the value of i that gives the maximum value of r, you need to find the vertex of the parabola. Since the quadratic expression is already factorised, the quickest way to do this is to find the i-intercepts first.

Putting $r = 0$ gives

$$(20 + i)(30 - i) = 0.$$

Hence

$$i = -20 \quad \text{or} \quad i = 30,$$

so the i-intercepts are -20 and 30.

The value halfway between the i-intercepts is

$$\frac{-20 + 30}{2} = \frac{10}{2} = 5.$$

Substituting $i = 5$ into the equation of the parabola gives

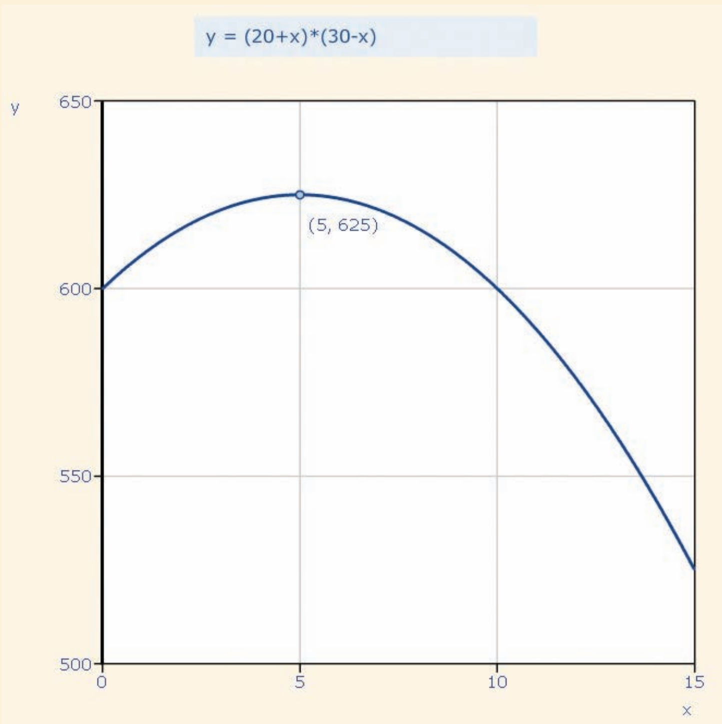
$$r = (20 + i)(30 - i) = 25 \times 25 = 625.$$

So the vertex of the parabola is $(5, 625)$.

State a conclusion in the context of the problem.

The boatman can maximise his revenue by increasing his price by £5, to £25. Then his revenue is likely to be £625.

(Check: the vertex of the Graphplotter graph below is about $(5, 625)$.)



The Graphplotter graph that was used to check the answer to the example above was obtained by using 'Custom function', which is available from the

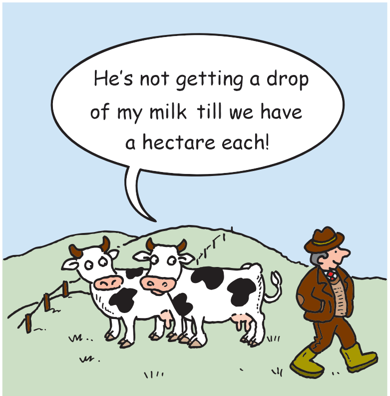
drop-down list of equations. There is information about using Custom function in Section 2 of the MU123 Guide, and also on the Graphplotter Help page (press the orange 'Help' button at the top right of Graphplotter).

In Example 12 a quadratic model was used to help to maximise a financial yield. Models like this can also be used to help to maximise other types of yields, such as agricultural yields.

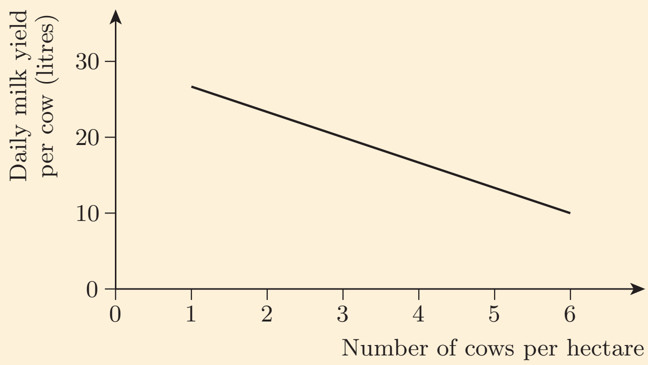
For example, consider a farmer who would like to maximise the yield from a fruit orchard. She may be able to increase the yield by increasing the number of trees, but if she increases the number of trees too much, then overcrowding will cause a large drop in the yield per tree, and the overall yield will fall.

Similarly, a dairy farmer may be able to increase the milk yield of his farm by increasing the number of cows in his fields, but if he increases the number of cows too much, then overcrowding will tend to decrease the milk yield per cow, causing the overall yield to fall. This is the scenario in the activity below.

Activity 25 *Maximising milk yield*



A dairy farmer wants to work out the number of cows that he should stock per hectare of grazing land to maximise the milk yield. The graph below, which he has seen in an agricultural magazine, gives the daily milk yield per cow that can be expected for different numbers of cows per hectare.



If the number of cows per hectare is denoted by n, and the daily milk yield per cow in litres is denoted by m, then the equation of the line on the graph is

$$m = -\tfrac{10}{3}n + 30.$$

(This equation can be found by using the methods that you learned in Unit 6.)

(a) Let y be the daily milk yield per hectare of grazing land, in litres. Find a formula for y in terms of n. You need to think about the number of cows on each hectare of land, and the daily milk yield from each of these cows.

(b) Hence determine the maximum daily milk yield per hectare, and the number of cows per hectare that will produce this yield.

(c) How many cows should the farmer stock on an 8-hectare field?

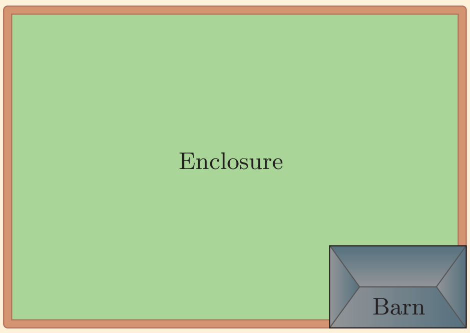
5.3 Maximising areas

In this subsection you will see two problems that involve maximising areas. You will see that it takes quite a lot of creative thought to solve the first problem, and it involves a number of different ideas that you have learned in the module. The second problem uses similar ideas, but it is a bit more straightforward, and you are asked to try to solve it yourself.

Example 13 *Maximising an area*

Tutorial clip

A farmer wants to construct an L-shaped enclosure with a boundary made up of fencing and two sides of a barn, as shown below. The barn is 8 m by 12 m, and the farmer has 100 m of fencing. What is the maximum area of the enclosure, and what are the dimensions of the enclosure with this maximum area?



Solution

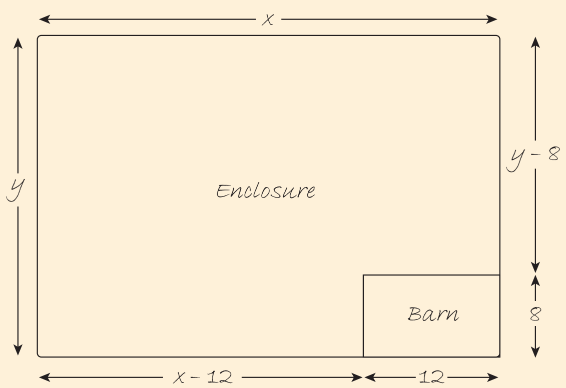
First choose the variables.

The quantity to be maximised is the area of the enclosure, so denote that by $A\,\mathrm{m}^2$.

The quantity that makes the area increase and decrease is the length of the enclosure, so denote that by x m.

You might think that it is not just the length of the enclosure that makes the area increase and decrease, but also its width. However, because there is only 100 m of fencing, the greater the length, the smaller the width. So the area is determined by only one variable. This one variable could be either the length or the width.

Now you need to find a formula for A in terms of x. To work out the area in terms of x you need to know the width of the enclosure, as well as its length, in terms of x. This is not obvious, so denote the width by y and use an equation to work it out. It helps to begin by drawing and labelling a diagram.



Since the total length of the fence is 100 m, we have

$$(x-12)+y+x+(y-8)=100.$$

Simplify this equation and make y the subject.

$$2x+2y-20=100$$
$$2y=120-2x$$
$$y=60-x$$

So the width of the field is $(60-x)$ m.

Now you can find a formula for A in terms of x.

The total area of the field can be found by subtracting the area of the barn from the area of the larger rectangle. This gives

$$A=x(60-x)-8\times 12,$$

which can be simplified to

$$\begin{aligned} A &= 60x-x^2-96 \\ &= -x^2+60x-96. \end{aligned}$$

The next step is to find the vertex of the parabola that is the graph of this equation. Let's do that by completing the square.

Completing the square gives

$$\begin{aligned} A &= -x^2+60x-96 \\ &= -(x^2-60x)-96 \\ &= -((x-30)^2-30^2)-96 \\ &= -((x-30)^2-900)-96 \\ &= -(x-30)^2+900-96 \\ &= -(x-30)^2+804. \end{aligned}$$

So the vertex is $(30, 804)$.

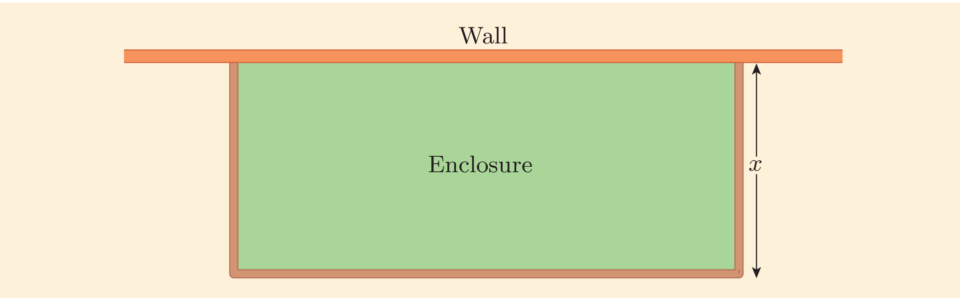
State a conclusion in the context of the question.

Hence the maximum area of the enclosure is $804\,\mathrm{m}^2$, and this occurs when the length of the enclosure is 30 m and the width is $(60-30)\,\mathrm{m}=30\,\mathrm{m}$.

Here is the activity for you to try.

Activity 26 *Maximising an area*

A farmer wants to make a rectangular enclosure next to an existing wall, using 120 m of fencing. Let the area of the enclosure be $A\,\mathrm{m}^2$, and let its width, as shown in the diagram below, be x m.



(a) Find an expression for the length of the enclosure in terms of x.

(b) Find a formula for A in terms of x.

(c) Hence find the maximum area of the enclosure, and the length and width that give this maximum area.

In this section you have seen some ideas for solving maximisation problems. Similar ideas can be used to solve **minimisation problems**, which are relevant for real-life situations that are modelled by u-shaped parabolas rather than n-shaped ones.

To end this section, here is a summary of the main steps used to solve a maximisation problem.

Strategy *To solve a maximisation problem*

1. Identify the quantity to be maximised and the quantity that it depends on, and denote each quantity by a variable.
2. Find a formula for the variable to be maximised in terms of the variable that it depends on.
3. If this gives a quadratic function, then find the vertex of its graph.

In this unit you have seen some quadratic models and their uses. You have learned about the graphs of quadratic functions, and how to sketch them, and you have also learned some new techniques for solving quadratic equations.

Learning checklist

After studying this unit, you should be able to:

- understand how some real-life situations can be modelled by quadratic functions
- understand the shapes and positions of the graphs of quadratic functions
- find the intercepts, axis of symmetry and vertex of the graph of a quadratic function from its equation
- sketch the graph of a quadratic function
- complete the square in quadratic expressions
- solve a quadratic equation by using the quadratic formula, by completing the square or by using an accurate graph
- determine the number of solutions of a quadratic equation from its coefficients
- construct simple quadratic models to describe real-life situations
- solve maximisation problems involving quadratic functions
- understand and apply Galileo's models for the motion of objects in free fall and projectiles launched horizontally or vertically.

Solutions and comments on Activities

Activity 1

Substituting $d = 26$ into the free-fall equation $d = 4.9t^2$ gives

$$26 = 4.9t^2.$$

Solving this equation gives

$$4.9t^2 = 26$$

$$t^2 = \frac{26}{4.9}$$

$$t = \sqrt{\frac{26}{4.9}} = 2.3 \text{ (to 1 d.p.)}.$$

(The positive square root is taken because the negative root does not make sense in this context.)

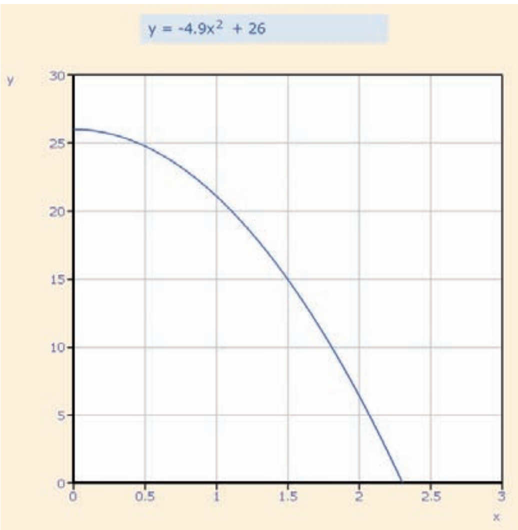
So the ball reaches the ground after about 2.3 seconds.

Activity 2

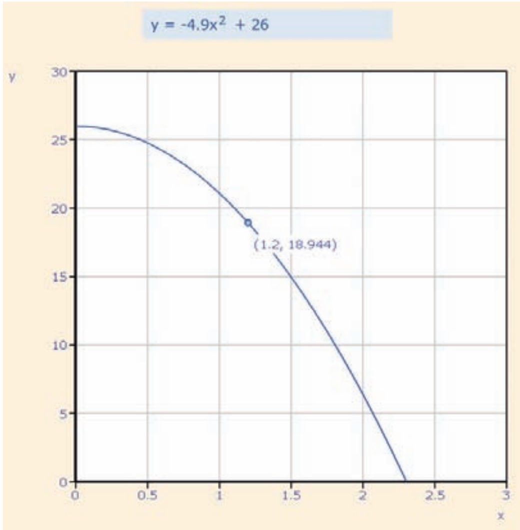
(a) From Activity 1, you know that the ball takes less than 3 seconds to fall to the ground, so a reasonable range for the x-axis is 0 to 3.

The ball falls from a height of 26 m, so a reasonable range for the y-axis is 0 to 30.

The Graphplotter graph is shown below.



(b) The height of the ball above the ground after 1.2 seconds is approximately 19 m, as shown in the following Graphplotter graph.



Activity 3

(a) The time t seconds for the marble to hit the floor is given by the free-fall equation

$$d = 4.9t^2,$$

where $d = 0.8$, measured in metres.

Substituting into the equation and solving it gives

$$0.8 = 4.9t^2$$

$$4.9t^2 = 0.8$$

$$t^2 = \frac{0.8}{4.9}$$

$$t = \sqrt{\frac{0.8}{4.9}} = 0.4040\ldots = 0.40 \text{ (to 2 s.f.)}.$$

(The positive square root is taken because the negative one does not make sense in this context.)

So the marble will hit the floor after about 0.40 s.

(b) The horizontal distance that the marble will travel before hitting the floor is given by the equation

$$d = st,$$

where $s = 0.5$ and, from part (a), $t = 0.4040\ldots$, where the units are metres per second and seconds, respectively.

Substituting into the equation gives

$$d = 0.5 \times 0.4040\ldots = 0.20 \text{ (to 2 s.f.)}.$$

So the marble will travel about 0.20 m before hitting the floor.

Activity 4

For example, one of the speeds given in Table 2 is 40 mph. Substituting $s = 40$ into formula (5), which is $B = \frac{1}{20}s^2$, gives

$$B = \tfrac{1}{20} \times 40^2 = 80.$$

So, according to the formula, a speed of 40 mph corresponds to a braking distance of 80 feet, which accords with the numbers given in Table 2.

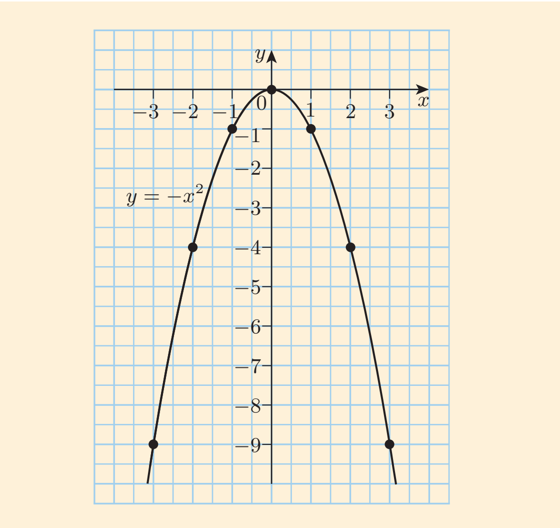
Activity 5

(a) A table of values for the equation $y = -x^2$ is given below.

x	-3	-2	-1	0	1	2	3
y	-9	-4	-1	0	-1	-4	-9

(For example, substituting $x = -3$ into the equation gives $y = -(-3)^2 = -9$.)

The resulting graph is shown below.



(b) The graph of $y = -x^2$ is a mirror image of the graph of $y = x^2$, reflected in the x-axis.

Activity 6

(a) Increasing the value of a within the positive values seems to make the parabola more narrow, while decreasing the value of a within the positive values seems to make it wider. The axis of symmetry is always the y-axis.

(b) Pairs of values of a that are negatives of each other give graphs that are reflections of each other in the x-axis, as expected.

So, as expected, increasing the *size* of a within the negative values seems to make the parabola narrower, while decreasing the size of a within the negative values seems to make it wider.

Activity 7

There are comments in the text after this activity.

Activity 8

(b) Changing c from 0 to 1 moves the graph up by 1 unit. The new graph crosses the y-axis at $(0, 1)$.

(c) The graph of $y = x^2 + c$ seems to be exactly the same as the graph of $y = x^2$, but shifted up or down the y-axis. It crosses the y-axis at $(0, c)$ (whether c is positive or negative).

(d) In general, whatever the values of a, b and c, the graph of $y = ax^2 + bx + c$ seems to be exactly the same as the graph of $y = ax^2 + bx$, but shifted vertically up or down, so that it crosses the y-axis at $(0, c)$.

Activity 9

The equation is $y = x^2 + 5x - 6$.

The coefficient of x^2 is positive, so the graph is u-shaped.

Putting $x = 0$ gives $y = -6$, so the y-intercept is -6.

Putting $y = 0$ gives

$$x^2 + 5x - 6 = 0$$
$$(x - 1)(x + 6) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x + 6 = 0$$
$$x = 1 \quad \text{or} \quad x = -6.$$

So the x-intercepts are 1 and -6.

The value halfway between the x-intercepts is

$$\frac{1 + (-6)}{2} = \frac{-5}{2} = -2.5.$$

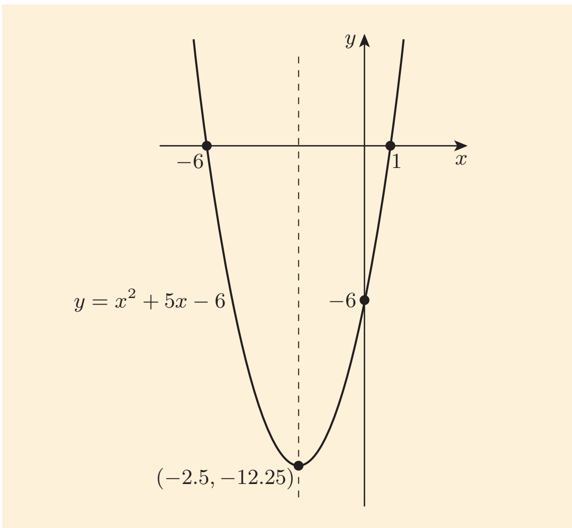
So the axis of symmetry is $x = -2.5$.

Substituting $x = -2.5$ into the equation of the parabola gives

$$\begin{aligned} y &= x^2 + 5x - 6 \\ &= (-2.5)^2 + 5 \times (-2.5) - 6 \\ &= -12.25. \end{aligned}$$

So the vertex is $(-2.5, -12.25)$.

A sketch of the graph is shown below.



Activity 10

The equation is $y = 9x^2 - 6x + 1$.

The coefficient of x^2 is positive, so the graph is u-shaped.

Putting $x = 0$ gives $y = 1$, so the y-intercept is 1.

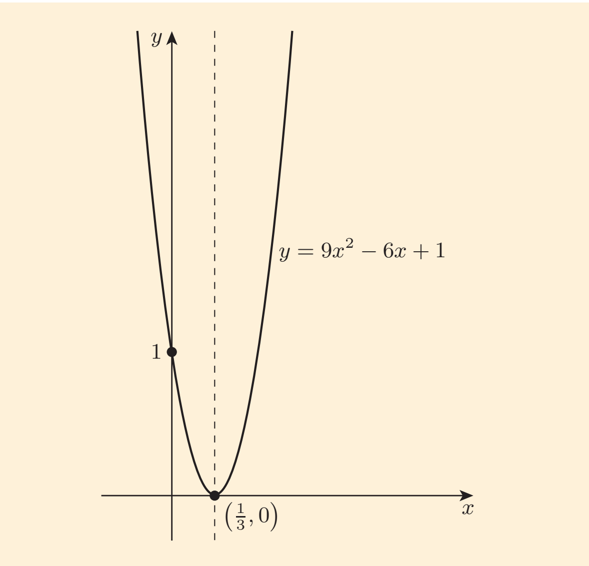
Putting $y = 0$ gives

$$9x^2 - 6x + 1 = 0$$
$$(3x - 1)(3x - 1) = 0$$
$$3x - 1 = 0$$
$$x = \tfrac{1}{3}.$$

So the only x-intercept is $\frac{1}{3}$.

Therefore the axis of symmetry is $x = \frac{1}{3}$, and the vertex is $\left(\frac{1}{3}, 0\right)$.

A sketch of the graph is shown below.



(If you feel uncomfortable basing your sketch on just two points, then you can calculate a third point on the parabola. For example, substituting $x = 1$ into the equation gives

$$y = 9 \times 1^2 - 6 \times 1 + 1 = 4,$$

so you can plot the point $(1, 4)$ to make the shape of the parabola clearer.)

Activity 11

The equation is $y = x^2 + 2x + 3$.

The coefficient of x^2 is positive, so the graph is u-shaped.

Putting $x = 0$ gives $y = 3$, so the y-intercept is 3.

The equation of the axis of symmetry is $x = -b/(2a)$, where $a = 1$ and $b = 2$, so it is

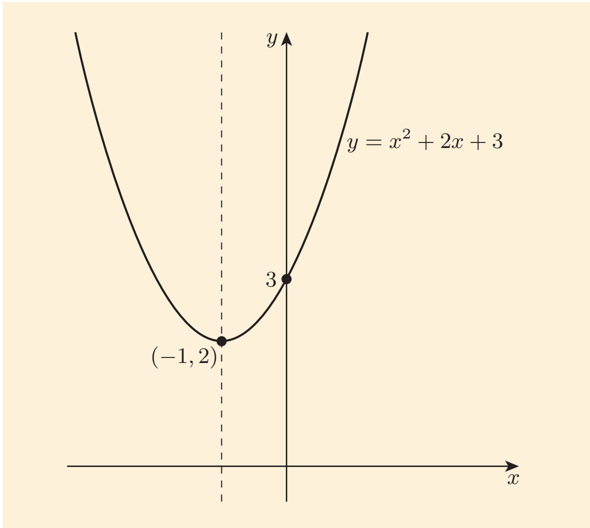
$$x = -\frac{2}{2 \times 1}; \quad \text{that is,} \quad x = -1.$$

Substituting $x = -1$ into the equation of the parabola gives

$$\begin{aligned} y &= x^2 + 2x + 3 \\ &= (-1)^2 + 2 \times (-1) + 3 \\ &= 1 - 2 + 3 \\ &= 2. \end{aligned}$$

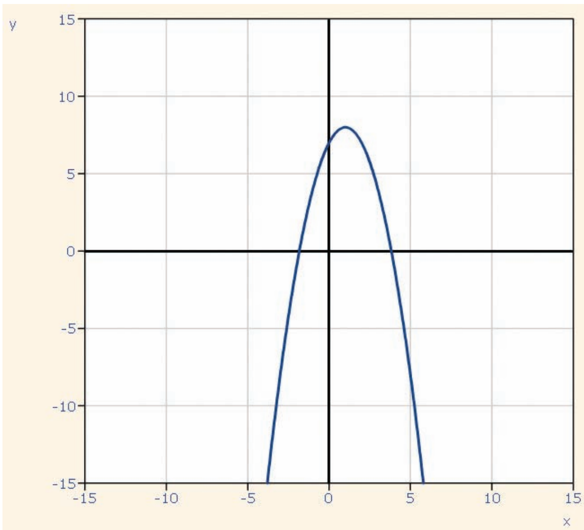
So the vertex is $(-1, 2)$.

A sketch of the graph is shown below.



Activity 12

(a) The graph of $y = -x^2 + 2x + 7$ is shown below.



(b) The solutions of the equation $-x^2 + 2x + 7 = 0$ are $x = -1.83$ and $x = 3.83$ (to 2 d.p.).

Activity 13

You should have found that the point with x-coordinate 1.10 has a y-coordinate greater than 20, while the point with x-coordinate 1.11 has a y-coordinate less than 20. (The x-coordinate 1.10 is displayed as 1.1 even if you have zoomed in enough for x-coordinates to be displayed to two decimal places, because Graphplotter does not display trailing zeros.)

Hence the point whose y-coordinate is exactly 20 has x-coordinate 1.1, to one decimal place.

So the ball takes approximately 1.1 seconds to fall to a height of 20 m.

Activity 14

(a) The equation is

$$x^2 + 6x + 1 = 0,$$

so $a = 1$, $b = 6$ and $c = 1$.

The quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{-6 \pm \sqrt{36 - 4}}{2} \\ &= \frac{-6 \pm \sqrt{32}}{2} \\ &= \frac{-6 \pm 4\sqrt{2}}{2} \\ &= -3 \pm 2\sqrt{2}. \end{aligned}$$

So the solutions are $x = -3 + 2\sqrt{2}$ and $x = -3 - 2\sqrt{2}$.

(b) The equation is

$$3x^2 - 8x - 2 = 0,$$

so $a = 3$, $b = -8$ and $c = -2$.

The quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 3 \times (-2)}}{2 \times 3} \\ &= \frac{8 \pm \sqrt{64 + 24}}{6} \\ &= \frac{8 \pm \sqrt{88}}{6} \\ &= \frac{8 \pm 2\sqrt{22}}{6} \\ &= \frac{4 \pm \sqrt{22}}{3}. \end{aligned}$$

So the solutions are $x = \dfrac{4 + \sqrt{22}}{3}$ and $x = \dfrac{4 - \sqrt{22}}{3}$.

Activity 15

The equation is

$$-x^2 = -x - \tfrac{3}{2}.$$

Clearing the fraction gives

$$-2x^2 = -2x - 3.$$

This equation can be rearranged as follows.

$$\begin{aligned} 0 &= 2x^2 - 2x - 3 \\ 2x^2 - 2x - 3 &= 0 \end{aligned}$$

So $a = 2$, $b = -2$ and $c = -3$.

The quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 2 \times (-3)}}{2 \times 2} \\ &= \frac{2 \pm \sqrt{4 + 24}}{4} \\ &= \frac{2 \pm \sqrt{28}}{4} \\ &= \frac{2 \pm 2\sqrt{7}}{4} \\ &= \frac{1 \pm \sqrt{7}}{2}. \end{aligned}$$

So the solutions are $x = \dfrac{1 + \sqrt{7}}{2}$ and $x = \dfrac{1 - \sqrt{7}}{2}$.

Activity 16

(a) The width of the photograph is 12 inches, and there are x inches of white border on each side, so the width of the backboard is $12 + 2x$ inches. Similarly, the height of the photograph is 18 inches, and there are x inches of white border both above and below, so the height of the backboard is $18 + 2x$ inches.

(b) The area of the photograph is

$$\text{width} \times \text{height} = 12 \times 18 = 216\,\text{in}^2.$$

The area of the backboard is 150% of the area of the photograph, so its area is

$$150\% \text{ of } 216 = 1.5 \times 216 = 324\,\text{in}^2.$$

(c) The area of the backboard in terms of x is

$$\begin{aligned} \text{width} \times \text{height} &= (12 + 2x)(18 + 2x) \\ &= 216 + 24x + 36x + 4x^2 \\ &= 4x^2 + 60x + 216, \end{aligned}$$

where the units are square inches.

(d) Using the answers to parts (b) and (c) gives

$$4x^2 + 60x + 216 = 324.$$

This equation can be rearranged as follows.

$$4x^2 + 60x - 108 = 0$$
$$x^2 + 15x - 27 = 0$$

(e) The equation in part (d) cannot easily be factorised, so we use the quadratic formula.

We have $a = 1$, $b = 15$ and $c = -27$.

Using the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-15 \pm \sqrt{15^2 - 4 \times 1 \times (-27)}}{2 \times 1} \\ &= \frac{-15 \pm \sqrt{225 + 108}}{2} \\ &= \frac{-15 \pm \sqrt{333}}{2} \\ &= \frac{-15 \pm 3\sqrt{37}}{2} \\ &= \frac{-15 + 3\sqrt{37}}{2} \quad \text{or} \quad \frac{-15 - 3\sqrt{37}}{2} \\ &= 1.624\ldots \quad \text{or} \quad -16.624\ldots. \end{aligned}$$

Only the positive solution makes sense in the context of the problem. So the width of the white border is 1.6 inches, to the nearest tenth of an inch.

The width of the backboard is

$$12 + 2 \times 1.624\ldots = 15.2\,\text{in},$$

and its height is

$$18 + 2 \times 1.624\ldots = 21.2\,\text{in},$$

to the nearest tenth of an inch.

Activity 17

(a) The equation is

$$9x^2 + 30x + 25 = 0,$$

so $a = 9$, $b = 30$ and $c = 25$.

The discriminant is

$$\begin{aligned} b^2 - 4ac &= 30^2 - 4 \times 9 \times 25 \\ &= 900 - 900 \\ &= 0. \end{aligned}$$

Since the discriminant is 0, there is one solution.

The equation can be solved by factorising, as below. Since there is only one solution, the two linear expressions in the factorisation must be the same (or one must be a multiple of the other – that is, it must be the other multiplied through by some number).

$$9x^2 + 30x + 25 = 0$$
$$(3x + 5)(3x + 5) = 0$$
$$(3x + 5)^2 = 0$$
$$3x + 5 = 0$$
$$x = -\tfrac{5}{3}$$

(b) The equation is

$$x^2 - 4x - 2 = 0,$$

so $a = 1$, $b = -4$ and $c = -2$.

The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4 \times 1 \times (-2) \\ &= 16 - (-8) \\ &= 24. \end{aligned}$$

Since the discriminant is greater than zero, there are two solutions.

The equation cannot be easily factorised, so we solve it by using the quadratic formula. This gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{24}}{2 \times 1} \\ &= \frac{4 \pm 2\sqrt{6}}{2} \\ &= 2 \pm \sqrt{6}. \end{aligned}$$

The solutions are $x = 2 + \sqrt{6}$ and $x = 2 - \sqrt{6}$.

(The value of $b^2 - 4ac$ was already worked out before the quadratic formula was used, so this value was just substituted in, instead of working it out again.)

(c) The equation can be rearranged as follows.

$$-3x^2 + 5x = 4$$
$$3x^2 - 5x + 4 = 0$$

So $a = 3$, $b = -5$ and $c = 4$.

The discriminant is

$$b^2 - 4ac = (-5)^2 - 4 \times 3 \times 4 = 25 - 48 = -23.$$

Since the discriminant is less than zero, there are no solutions.

Activity 18

The height h metres of the rocket after time t seconds is given by the formula

$$h = -\tfrac{1}{2}gt^2 + v_0t + h_0,$$

with $h_0 = 155$, $v_0 = 49$ and $g = 9.8$. So

$$h = -4.9t^2 + 49t + 155.$$

The rocket returns to the ground when $h = 0$, so we must solve the equation

$$-4.9t^2 + 49t + 155 = 0.$$

Multiplying through by -1 gives

$$4.9t^2 - 49t - 155 = 0.$$

We use the quadratic formula, with $a = 4.9$, $b = -49$ and $c = -155$. This gives

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-49) \pm \sqrt{(-49)^2 - 4 \times 4.9 \times (-155)}}{2 \times 4.9} \\ &= \frac{49 \pm \sqrt{2401 + 3038}}{9.8} \\ &= \frac{49 \pm \sqrt{5439}}{9.8} \\ &= -2.5 \quad \text{or} \quad 12.5 \quad \text{(to 1 d.p.)}. \end{aligned}$$

The negative solution does not make sense in this context, so we disregard it. So the time taken for the rocket to reach the ground is about 12.5 seconds.

Activity 19

(a) The parabola is u-shaped.

Its vertex is $(-1, 5)$.

(b) The parabola is n-shaped.

Its vertex is $(-3, 7)$.

(c) The parabola is u-shaped.

Its vertex is $(1, -4)$.

(d) The parabola is n-shaped.

Its vertex is $\left(-\frac{1}{2}, -1\right)$.

(e) The parabola is u-shaped.

Its vertex is $(0, 3)$.

(f) The parabola is u-shaped.

Its vertex is $(2, 0)$.

Activity 20

(a) $$\begin{aligned} x^2 + 16x &= (x+8)^2 - 8^2 \\ &= (x+8)^2 - 64 \end{aligned}$$

(Check: $$\begin{aligned} (x+8)^2 - 64 &= (x+8)(x+8) - 64 \\ &= x^2 + 16x + 64 - 64 \\ &= x^2 + 16x.) \end{aligned}$$

(b) $$\begin{aligned} x^2 - 12x &= (x-6)^2 - (-6)^2 \\ &= (x-6)^2 - 36 \end{aligned}$$

(Check: $$\begin{aligned} (x-6)^2 - 36 &= (x-6)(x-6) - 36 \\ &= x^2 - 12x + 36 - 36 \\ &= x^2 - 12x.) \end{aligned}$$

(c) $$\begin{aligned} t^2 - 2t &= (t-1)^2 - (-1)^2 \\ &= (t-1)^2 - 1 \end{aligned}$$

(Check: $$\begin{aligned} (t-1)^2 - 1 &= (t-1)(t-1) - 1 \\ &= t^2 - 2t + 1 - 1 \\ &= t^2 - 2t.) \end{aligned}$$

(d) $$\begin{aligned} x^2 + 3x &= \left(x + \tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} \end{aligned}$$

(Check: $$\begin{aligned} \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} &= \left(x + \tfrac{3}{2}\right)\left(x + \tfrac{3}{2}\right) - \tfrac{9}{4} \\ &= x^2 + \tfrac{3}{2}x + \tfrac{3}{2}x + \tfrac{9}{4} - \tfrac{9}{4} \\ &= x^2 + 3x.) \end{aligned}$$

Activity 21

(a) $$\begin{aligned} x^2 + 6x - 3 &= (x+3)^2 - 9 - 3 \\ &= (x+3)^2 - 12 \end{aligned}$$

(Check: $$\begin{aligned} (x+3)^2 - 12 &= x^2 + 6x + 9 - 12 \\ &= x^2 + 6x - 3.) \end{aligned}$$

(b) $$\begin{aligned} x^2 - 4x + 9 &= (x-2)^2 - 4 + 9 \\ &= (x-2)^2 + 5 \end{aligned}$$

(Check: $$\begin{aligned} (x-2)^2 + 5 &= x^2 - 4x + 4 + 5 \\ &= x^2 - 4x + 9.) \end{aligned}$$

(c) $$\begin{aligned} p^2 - 12p - 5 &= (p-6)^2 - 36 - 5 \\ &= (p-6)^2 - 41 \end{aligned}$$

(Check: $$\begin{aligned} (p-6)^2 - 41 &= p^2 - 12p + 36 - 41 \\ &= p^2 - 12p - 5.) \end{aligned}$$

(d) $$\begin{aligned} x^2 + x + 1 &= (x + \tfrac{1}{2})^2 - \tfrac{1}{4} + 1 \\ &= (x + \tfrac{1}{2})^2 + \tfrac{3}{4} \end{aligned}$$

(Check: $$\begin{aligned} (x + \tfrac{1}{2})^2 + \tfrac{3}{4} &= x^2 + x + \tfrac{1}{4} + \tfrac{3}{4} \\ &= x^2 + x + 1.) \end{aligned}$$

Activity 22

(a) $$\begin{aligned} x^2 + 6x - 5 &= 0 \\ (x+3)^2 - 9 - 5 &= 0 \\ (x+3)^2 - 14 &= 0 \\ (x+3)^2 &= 14 \\ x + 3 &= \pm\sqrt{14} \\ x &= -3 \pm \sqrt{14} \end{aligned}$$

The solutions are $x = -3 + \sqrt{14}$ and $x = -3 - \sqrt{14}$.

(b) $2x^2 - 12x - 5 = 0$

$$\begin{aligned} x^2 - 6x - \tfrac{5}{2} &= 0 \\ (x-3)^2 - 9 - \tfrac{5}{2} &= 0 \\ (x-3)^2 - \tfrac{18}{2} - \tfrac{5}{2} &= 0 \\ (x-3)^2 - \tfrac{23}{2} &= 0 \\ (x-3)^2 &= \tfrac{23}{2} \\ x - 3 &= \pm\sqrt{\tfrac{23}{2}} \\ x &= 3 \pm \sqrt{\tfrac{23}{2}} \end{aligned}$$

The solutions are $x = 3 + \sqrt{\frac{23}{2}}$ and $x = 3 - \sqrt{\frac{23}{2}}$.

Activity 23

(a) $$\begin{aligned} 2x^2 - 4x - 1 &= 2(x^2 - 2x) - 1 \\ &= 2((x-1)^2 - 1) - 1 \\ &= 2(x-1)^2 - 2 - 1 \\ &= 2(x-1)^2 - 3 \end{aligned}$$

(Check: $$\begin{aligned} 2(x-1)^2 - 3 &= 2(x^2 - 2x + 1) - 3 \\ &= 2x^2 - 4x + 2 - 3 \\ &= 2x^2 - 4x - 1.) \end{aligned}$$

The vertex of the parabola with equation $y = 2x^2 - 4x - 1$ is $(1, -3)$.

(b) $$\begin{aligned} -x^2 - 8x - 18 &= -(x^2 + 8x) - 18 \\ &= -((x+4)^2 - 16) - 18 \\ &= -(x+4)^2 + 16 - 18 \\ &= -(x+4)^2 - 2 \end{aligned}$$

(Check: $$\begin{aligned} -(x+4)^2 - 2 &= -(x^2 + 8x + 16) - 2 \\ &= -x^2 - 8x - 16 - 2 \\ &= -x^2 - 8x - 18.) \end{aligned}$$

The vertex of the parabola with equation $y = -x^2 - 8x - 18$ is $(-4, -2)$.

Activity 24

The motion of the ball is modelled by the equation

$$h = -\tfrac{1}{2}gt^2 + v_0 t + h_0$$

with $h_0 = 1.6$, $v_0 = 15$ and $g = 9.8$. So its height in metres after t seconds is given by

$$h = -4.9t^2 + 15t + 1.6.$$

The maximum value of h is at the vertex of this parabola.

The t-coordinate of the vertex is given by

$$t = -\frac{b}{2a},$$

where $a = -4.9$ and $b = 15$, so it is

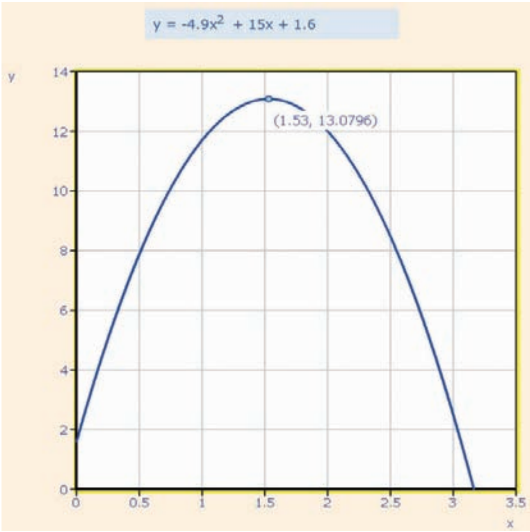
$$t = -\frac{15}{2 \times (-4.9)} = 1.5306\ldots.$$

Substituting $t = 1.5306\ldots$ into the equation of the parabola gives

$$\begin{aligned} h &= -4.9t^2 + 15t + 1.6 \\ &= -4.9 \times (1.5306\ldots)^2 + 15 \times (1.5306\ldots) + 1.6 \\ &= 13.079\ldots \\ &= 13 \text{ (to the nearest whole number).} \end{aligned}$$

So the ball reaches a height of approximately 13 m.

(This answer can be checked using the Graphplotter graph below.)



Activity 25

(a) There are n cows per hectare, and each cow produces m litres of milk per day, where m is given by the formula in the question. So y, the total daily milk yield per hectare, in litres, is given by

$$y = n \times m,$$

that is,

$$y = n\left(-\tfrac{10}{3}n + 30\right).$$

(b) We have to find the vertex of the parabola that is the graph of the equation found in part (a). We do this by finding the n-intercepts first, since the expression on the right-hand side of the equation is already factorised.

Putting $y = 0$ gives

$$n\left(-\tfrac{10}{3}n + 30\right) = 0,$$

so

$$n = 0 \quad \text{or} \quad -\tfrac{10}{3}n + 30 = 0.$$

Solving the linear equation on the right gives

$$\begin{aligned} -\tfrac{10}{3}n &= -30 \\ \tfrac{10}{3}n &= 30 \\ n &= 30 \times \tfrac{3}{10} \\ n &= 9. \end{aligned}$$

So the n-intercepts are 0 and 9.

The value halfway between the n-intercepts is 4.5.

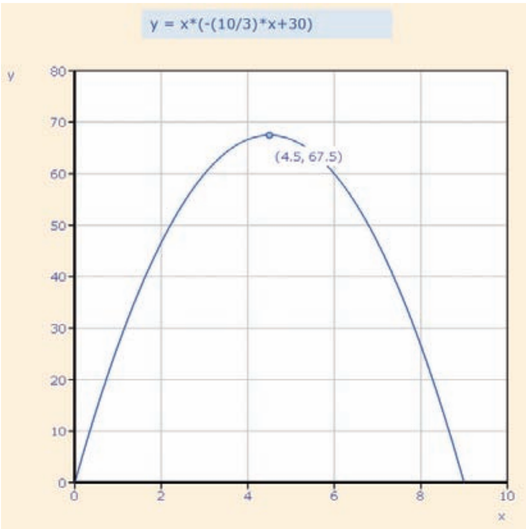
Substituting $n = 4.5$ into the equation of the parabola gives

$$\begin{aligned} y &= 4.5 \times \left(-\tfrac{10}{3} \times 4.5 + 30\right) \\ &= 4.5 \times (-15 + 30) \\ &= 67.5. \end{aligned}$$

So the vertex of the parabola is $(4.5, 67.5)$.

Hence the farmer can maximise the milk yield by stocking 4.5 cows per hectare. (Though the number of cows in a field will need to be rounded to a whole number, of course!) At a stocking rate of 4.5 cows per hectare, the daily milk yield will be 67.5 litres per hectare.

(This answer can be checked using the Graphplotter graph below, which was produced by using 'Custom function'.)



(c) The number of cows that the farmer should stock on an 8-hectare field is $8 \times 4.5 = 36$.

Activity 26

(a) The total length of fencing is 120 m, and two of the three sides of the enclosure are x m long. So the length of the third side is $(120 - 2x)$ m, and this is the length of the enclosure.

(b) The area of the enclosure is given by

$$A = x(120 - 2x).$$

(c) The formula found in part (b) is already factorised, so the quickest way to find the vertex is to find the x-intercepts first.

Putting $A = 0$ gives

$$x(120 - 2x) = 0,$$

so

$$x = 0 \quad \text{or} \quad x = \frac{120}{2} = 60.$$

So the x-intercepts are 0 and 60.

The value halfway between the x-intercepts is 30.

Substituting $x = 30$ into the equation of the parabola gives

$$\begin{aligned} A = 30(120 - 2 \times 30) &= 30(120 - 60) \\ &= 30 \times 60 = 1800. \end{aligned}$$

So the vertex is $(30, 1800)$.

Hence the maximum area of the enclosure is $1800\,\text{m}^2$, and this is achieved when the width is 30 m and the length is $(120 - 2 \times 30)\,\text{m} = 60\,\text{m}$.

(This answer can be checked using the GraphPlotter graph below, which was produced by using 'Custom function'.)